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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

What Price Isolation?

As in international politics, so in mathematics we have our isolationists. For them there must not be any educational entanglements. Let the psychologists and educationists fight it out among themselves. A plague on all their houses! And if we do not sell mathematics to either the progressives or the essentialists, soon enough both groups will starve to death and then mathematics and mathematicians will establish a natural protectorate over the bordering arts and sciences. With proper propaganda about our superiority, with every teacher a research worker (in the older, more genuine sense) and every student a prospective research worker, we shall keep on extending our frontiers until we shall have overrun the territories of the inferior branches of the family of human knowledge. And then,—when and if other worlds need to be made better, mathematicians will make them so.

To the non-isolationists the matter is not quite so simple. They see mathematics and mathematicians drawn into the educational turmoil despite the apparent protection of oceans of common sense and continents of tradition. Surrounded by and imbedded in networks of correlation, integration, application, and devitalization, real isolation seems to be out of question.

Larger issues than the preservation of, or domination by, mathematics are at stake. Some say that "democracy is on trial" and that its survival depends more on the education of its citizenry than any military equipment or front line defenses. Is mathematics essential to the right kind of education for a virile and enlightened democracy? Are there varieties of mathematics and mathematical education for the many diversified (intellectually and otherwise) strata which perforce form a democracy? For how long will mathematicians remain smugly aloof from concerted involvement in the educational turmoil of today? For how long will they stay content to let "others" muddle along? What price isolation?

Joseph Seidlin.

A Configuration Associated with a Tetrahedron

By J. W. PETERS
University of Illinois

1. *Introduction.* Associated with a plane triangle is a configuration which we shall call the Brocard-Beltrami configuration. The three Apollonian circles, C_1,C_2,C_3, of a triangle intersect in two points, called the isodynamic or Hessian points of the triangle. The centers of the Apollonian circles, k_1,k_2,k_3, lie on a line called the Lemoine Axis. The inverses of k_i in C_j and C_l $(i,j,l=1,2,3;\ i\neq j,l)$ determine two points called the Beltrami points. The isodynamic points are inverse points in the circumcircle of the triangle and also in the Lemoine Axis. The inverse of the Lemoine Axis in the circumcircle of the triangle is a circle on the circumcenter and known as the Second Brocard Circle. The inverses of the Beltrami points in the circumcircle are called the Brocard points of the triangle and the inverses of the Apollonian centers form a triangle called the Second Brocard Triangle. The Second Brocard Circle also passes through the Lemoine or symmedian point of the triangle, and the diameter of this circle is the distance between the circumcenter and the symmedian point. The circumcircle, the Second Brocard circle, and the Lemoine axis belong to a system of circles orthogonal to the Apollonian circles, called the Schoute coaxal system. The isodynamic points are null circles in the Schoute system.*

The essential fact to be derived from the discussion is that the group of twenty points, the vertices of the original triangle; the centers of the Apollonian circles; the counterpoints of the original triangle (these are the points in which the Apollonian circles intersect the circumcircle again); the vertices of the second Brocard triangle; the isodynamic points; the Brocard points; the Beltrami points; the circumcenter; and the point at infinity; constitute a configuration which is sent into itself by an inversion in the circumcircle or an inversion in any one of the Apollonian circles.

*The most complete account of the above discussed configuration is given in R. A. Johnson's book *Modern Geometry*, pp. 294-299. The reader should also see Morley and Morley's *Inversive Geometry*, pp. 77-79, where a reference to a paper by Beltrami is made. Note that Johnson makes no mention of Beltrami and on the other hand the Morleys failed to show that their Beltrami line is nothing more than the well known Lemoine axis.

Many attempts have been made to generalize the so-called Brocard geometry to four points in a plane, but writers were forced to restrict themselves to the harmonic quadrangle.* A more natural extension would be to the tetrahedron. P. Delens has discussed a number of topics in this connection in recent papers in Mathesis.† It is the intention of the author to investigate the problem from a different point of view than Delens, to arrive at explicit expressions for the points, planes, and spheres in the extended configuration, and to point out a number of properties of the configuration which have not been mentioned.

2. *The Inversive Theory of the Triangle and the Tetrahedron.* The bases of the Brocard-Beltrami configuration associated with the triangle are the circumcircle, the Apollonian circles, and the isodynamic points. These circles and points are inversive invariants of the triangle.‡ It would seem natural then to look among the inversive invariants of four points in three dimensional space to find a set of spheres and points which will furnish a basis for the Brocard-Beltrami configuration associated with a tetrahedron.

It was shown in the paper mentioned above that if 1, 2, 3, 4 are four points in S_3 and if 0 is a variable point, the following systems of equations represent spheres and points which are invariant under inversions;—

$$E_1 : (01)^2(23)(34)(42) = (02)^2(34)(41)(13) = (03)^2(41)(12)(24) = (04)^2(12)(23)(31),$$

and

$$E_2 : (01)^2(23)(34)(42) \pm (02)^2(34)(41)(13) \pm (03)^2(41)(12)(24) \pm (04)(12)(23)(31) = 0,$$

where (ij) represents the distance between the points i and j. The system E_1 represents six spheres which intersect in two points. Following the notation of Delens, we shall call these spheres the *Apollonian Spheres of the First Kind*, and their points of intersection the *Isodynamic Points* of the tetrahedron. In the system E_2 the use of all positive signs gives an imaginary sphere. Using one minus sign, $+++-$, $++-+$, $+-++$, $-+++$, four points on the circumsphere, called the counterpoints of the tetrahedron, are obtained. Using two minus signs, $++--$, $+-+-$, $+--+$, three mutually orthogonal spheres, called the *Apollonian spheres of the second kind*, are obtained.

*R. A. Johnson, *Modern Geometry*, p. 301.
†*Mathesis*, Vol. 51, p. 119, p. 444; Vol. 52, p. 62.
‡*American Journal of Mathematics*, Vol. 51, p. 599.

These spheres intersect each other in the isodynamic points and intersect the circumsphere in six points called the Jacobian points. The relation between the four given points and the four counterpoints is mutual.

3. *The Apollonian Spheres.* Let the four vertices of the tetrahedron be represented by the end points of the vectors A_1,A_2,A_3,A_4, which have their initial point at the origin. Without loss of generality we may assume that the radius of the circumsphere is 1, so that $(A_1,A_1)=(A_2,A_2)=(A_3,A_3)=(A_4,A_4)=1$, where (A_i,A_i) represents the scalar product of the vector A_i with itself. If X represents the vector to a variable point and if we replace the products of the distances $(jk)(kl)(lj)$ by v_i, $(i,j,k,l=1,2,3,4)$ and $(i\neq j\neq k\neq l)$, the equations of the Apollonian spheres of the first kind may be written

$$v_1(X-A_1,X-A_1)=v_2(X-A_2,X-A_2)$$

$$=v_3(X-A_3,X-A_3)=v_4(X-A_4,X-A_4).$$

The sphere S_{ij} is given by the equation

$$v_i(X-A_i,X-A_i)=v_j(X-A_j,X-A_j),$$

has the center $C_{ij}=(v_iA_i-v_jA_j)/(v_i-v_j)$, and has the radius

$$r^2{}_{ij}=2v_iv_j\,[1-(A_i,A_j)]/(v_i-v_j)^2.$$

Using the above notation, we may write the system of equations representing the Apollonian spheres of the second kind as

$$v_1(X-A_1,X-A_1)\pm v_2(X-A_2,X-A_2)$$

$$\pm v_3(X-A_3,X-A_3)\pm v_4(X-A_4,X-A_4)=0,$$

where two minus signs are used. If the sign of the first term is kept positive, it will be convenient to represent the three spheres by S_2,S_3,S_4, where the subscript represents the number of the next positive term after the first. The centers of these spheres are:

$$C_2=(v_1A_1+v_2A_2-v_3A_3-v_4A_4)/(v_1+v_2-v_3-v_4),$$

$$C_3=(v_1A_1-v_2A_2+v_3A_3-v_4A_4)/(v_1-v_2+v_3-v_4),$$

$$C_4=(v_1A_1-v_2A_2-v_3A_3+v_4A_4)/(v_1-v_2-v_3+v_4),$$

and the corresponding radii will be designated by r_2,r_3,r_4 respectively. While it is possible to express these radii in terms of the v_i's and the A_i's, the results are involved and are of no particular use at the present time.

The points C_2,C_3,C_4, determine a plane, η. A point, P, is in this plane if real numbers k_1,k_2,k_3, can be found so that

$$P=\frac{k_1(v_1+v_2-v_3-v_4)C_2+k_2(v_1-v_2+v_3-v_4)C_3+k_3(v_1-v_2-v_3+v_4)C_4}{k_1(v_1+v_2-v_3-v_4)+k_2(v_1-v_2+v_3-v_4)+k_3(v_1-v_2-v_3+v_4)}$$

For the following values of the k_i, the Apollonian centers of the first kind are obtained:

k_1	k_2	k_3	P	k_1	k_2	k_3	B
0	1	1	C_{12}	0	1	−1	C_{34}
1	0	1	C_{13}	−1	0	1	C_{24}
1	1	0	C_{14}	1	−1	0	C_{23}

The values k_1,k_2,k_3, may be taken as homogeneous coordinates of the points in the plane, with C_2,C_3,C_4 as the reference triangle. The plane, η, shall be called the Plane of Lemoine.

It can be shown either by use of vectors or by means of the homogeneous coordinates that the Apollonian centers of the first kind lie by threes on four lines thus forming a complete quadrilateral in the plane, η, and that the Apollonian centers of the second kind are the diagonal points of this quadrilateral.

4. *The Beltrami Points.* Let the equation of the sphere S_i now be written as $(X-C_i,X-C_i)=r_i^2$. If Y and Z are inverse points in this sphere, then $(Y-C_i,Z-C_i)=r_i^2$ and the constants k_1 and k_2 can be found so that $Z=(k_1C_i+k_2Y)/(k_1+k_2)$. The last equation expresses that Y, Z, and C_i are on a line. The inverse of C_j in the sphere $S_i,(i\neq j)$, $(i,j=2,3,4)$, can now be found, and will be designated by $B_k(k=2,3,4;\ k\neq i\neq j)$. Under these conditions $k_1=1/r_i^2$ and k_2 $k_2=1/\{(C_i-C_j,C_i-C_j)-r_i^2\}$. But (C_i-C_j,C_i-C_j) is the square of the distance between the centers C_i and C_j and since the spheres are orthogonal, the denominator of k_2 may be written as r_i^2. Therefore $B_k=(r_j^2C_i+r_i^2C_j)/(r_i^2+r_j^2)$. From the symmetry of this equation it can be seen that the inverse of C_i in S_j is the same as the inverse of C_j in S_i. Hence, there are three points B_2,B_3, and B_4, which we shall call the Beltrami points of the tetrahedron. They, of course, lie in the plane of Lemoine.

The inverse of each Beltrami point, B_i, in the corresponding sphere, S_i, is the point C_1, given by the equation

$$C_1=\frac{r_3^2r_4^2C_2+r_4^2r_2^2C_3+r_2^2r_3^2C_4}{r_3^2r_4^2+r_4^2r_2^2+r_2^2r_3^2}\ .$$

It may be shown that C_1 is the radical center of the circles in which the Lemoine cut the Apollonian spheres of the second kind.

5. *The Brocard Points and the Brocard Sphere.* The circumsphere of the four points is given by the equation $(X,X)=1$. If X and Y are inverse points in this sphere, then $(X,Y)=1$ and $Y=kX$. The inverses of the Beltrami points, $B_i (i=2,3,4)$, are the points $\mathfrak{B}_i$ given by the equations:

$$\mathfrak{B}_i = B_i/(B_i,B_i) \qquad (i=2,3,4).$$

We shall call these points the Brocard points of the tetrahedron.

The inverses of the centers, C_{ij}, of the Apollonian spheres of the first kind are given by

$$\mathfrak{C}_{ij} = C_{ij}/(C_{ij},C_{ij}), \qquad (i,j=1,2,3,4 : i \neq j).$$

The inverses of the radical center, C_1, and the centers, C_i, of the Apollonian spheres of the second kind are given by

$$\mathfrak{C}_i = C_i/(C_i,C_i) \qquad (i=1,2,3,4).$$

The points $\mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4$; $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4$; and $\mathfrak{C}_{12}, \mathfrak{C}_{13}, \mathfrak{C}_{14}, \mathfrak{C}_{23}, \mathfrak{C}_{24}, \mathfrak{C}_{34}$, lie on a sphere which passes through the center of the circumsphere. Let us call this sphere the Brocard sphere.

6. *The Schoute System and the Isodynamic Points.* The circumsphere $(X,X)-1=0$ and the plane of Lemoine,

$$(X-C_2,X-C_3,X-C_4)=0,$$

from a pencil of spheres called the Schoute system. The equation is

$$(X,X)-1+2k\,(X-C_2,X-C_3,X-C_4)=0.$$

If we represent the vector $[C_2,C_3]+[C_3,C_4]+[C_4,C_2]$ by the vector V and the scalar triple product (C_2,C_3,C_4) by s, the equation of the Schoute system may be written as

$$(X,X)+2k\,(X,V)=1+2k\,s.$$

The center of any sphere of the system is $-kV$ and the radius is given by $r^2=1+2k\,s+k^2(V,V)$.

The null spheres in this system are determined by setting $r=0$, and finding the resulting values of k. In this case we have

$$k=\{-s\pm\sqrt{s^2-(V,V)}\}/(V,V).$$

*$[C_i, C_j]$ represents the vector product of C_i and C_j.

Hence, we have the two point spheres

$$I_1=\frac{s-\sqrt{s^2-(V,V)}}{(V,V)}\cdot V;\ I_2=\frac{s+\sqrt{s^2-(V,V)}}{(V,V)}\cdot V.$$

It is readily verified that $(I_1,I_2)=1$ and that I_1 is a scalar multiple of I_2. Therefore I_1 and I_2 are inverse points in the circumsphere.

The equation of the Lemoine plane may be written $(X,V)=s$. The midpoint of the line segment joining I_1 and I_2 lies in this plane, for $\frac{1}{2}(I_1+I_2)=s\ V/(V,V)$ and

$$\frac{s}{(V,V)}\cdot(V,V)=s.$$

Moreover I_1 and I_2 are parallel to the vector V, and since V is perpendicular to the plane of Lemoine, so must the line joining I_1 and I_2 be perpendicular to that plane. Hence I_1 and I_2 are reflections or inverse points in the plane of Lemoine.

The points I_1 and I_2 are both on each of the spheres S_2,S_3, and S_4. Hence they are the isodynamic points. The proof that I_1 is on S_2 follows. If I_1 is on S_2 it must satisfy the equation

$$(I_1,I_1)-2(C_2,I_1)+1=0.$$

When the value of I_1 is substituted in this equation, we get

$$\frac{2s^2-2s\sqrt{s^2-(V,V)}-(V,V)}{(V,V)}-2\ \frac{s-\sqrt{s^2-(V,V)}}{(V,V)}\cdot(C_2,V)+1=0.$$

Since $(C_2,V)=(C_2,[C_2,C_3]+[C_3,C_4]+[C_4,C_2])=(C_2,C_3,C_4)=s$, it is easily seen that the equation is satisfied.

The sphere of the Schoute system which passes through the circumcenter of the four points A_1,A_2,A_3,A_4, or the origin in this case, is that given by $k=-1/2s$. The equation of this sphere is

$$s(X,X)-(X,V)=0.$$

The Brocard points are on this sphere, so it is the Brocard sphere. The proof that $\mathfrak{B}_2$ is on the sphere follows. We must show that $\mathfrak{B}_2$ satisfies the equation $s(\mathfrak{B}_2,\mathfrak{B}_2)-(\mathfrak{B}_2,V)=0$. Since $\mathfrak{B}_2=B_2/(B_2,B_2)$, the problem is reduced to proving that $s-(B_2,V)=0$. Since

$$B_2=(r_4^2C_3+r_3^2C_4)/(r_4^2+r_3^2),$$

we must now show that $(r_4^2+r_3^2)s-(r_4^2C_3+r_3^2C_4,V)=0$. This readily follows since (C_3,V) and (C_4,V) are both equal to s.

7. *The Counterpoints and the Jacobian Points.* If in the system of spheres E_2, we choose those equations which contain only one minus sign, the four resulting equations will represent null spheres or points on the circumsphere of the four original points. These four points are called the counterpoints of the original set and were discussed in the paper on the inversive invariants of four points. Suffice it to say that they have the same Apollonian spheres of the first and second kinds, the same plane of Lemoine, the same Beltrami points, and the same Brocard points. We shall designate these points by A'_i where

$$A'_4=(v_1A_1+v_2A_2+v_3A_3-v_4A_4)/(v_1+v_2+v_3-v_4),$$

the subscript on A'_i being determined by the position of the minus sign in the particular equation of E_2.

The Jacobian points are the intersections of two Apollonian spheres of the second kind and the circumsphere. There are six such points. We shall be content to give the explicit form of just one pair, the intersections of S_2 and S_3 and the circumsphere.

$$J_4,J'_4=kC_2+lC_3+m[C_2,C_3]\ ;$$

where
$$k=\{(C_3,C_3)-(C_2,C_3)\}/\{(C_2,C_2)(C_3,C_4)-(C_2,C_3)^2\}$$
$$l=\{(C_2,C_2)-(C_2,C_3)\}/\{(C_2,C_2)(C_3,C_3)-(C_2,C_3)^2\}$$
$$m=\pm\sqrt{1-(C_2-C_3,C_2-C_3)}/\{(C_2,C_2)(C_3,C_3)-(C_2,C_3)^2\}.$$

For J_4 we choose the positive sign before the radical and for J'_4 we choose the negative sign. It is somewhat tedious to show that these values satisfy the equations of the three spheres.

8. *The Brocard-Beltrami Configuration for the Tetrahedron.* We shall call the configuration consisting of the following points and spheres, the Brocard-Beltrami configuration for the tetrahedron.

A. Twenty-four points: the four original points, the four counterpoints, the centers of the Apollonian spheres of the second kind, the radical center C_1, the inverses of this last group of four points on the circumsphere, the Beltrami points, the Brocard points, the circumcenter of the four original points, and the point at infinity.

B. Six null spheres called the Jacobian points.

C. Six Apollonian spheres of the first kind.

D. Five spheres of the Schoute system, namely, the circumsphere, the Brocard sphere, the plane Lemoine, and the two null spheres called the isodynamic points.

This configuration is invariant under inversion in any of the following spheres: the circumsphere and the Apollonian spheres of the second kind.

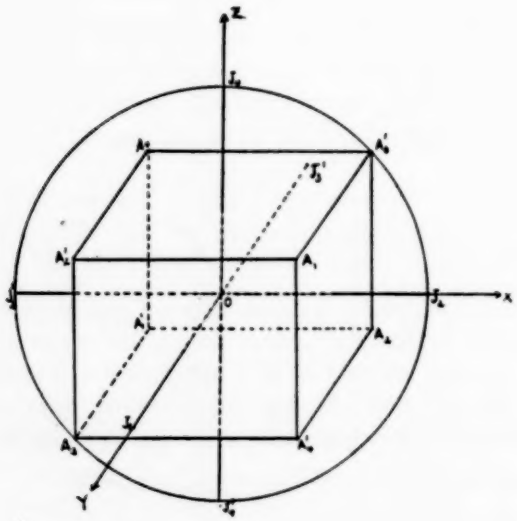
Proof:

A. Under inversion in the circumsphere, the original four points, the counterpoints, and the Jacobian points remain unaltered. The isodynamic points are interchanged, but each Apollonian sphere goes into itself, no matter which kind it is. The plane of Lemoine is sent into the Brocard sphere in such a way that

$$C_i \longrightarrow \mathfrak{C}_i, \quad C_{ij} \longrightarrow \mathfrak{C}_{ij}, \quad B_i \longrightarrow \mathfrak{B}_i, \quad \infty \longrightarrow 0;$$

and conversely.

B. In order to discover what happens under inversion in the Apollonian spheres of the second kind it will be convenient to keep in mind the canonical form of four points in the inversive theory, which is represented below.



The Apollonian spheres of the second kind, S_2, S_3, S_4, are represented by the coordinate planes YZ, XZ, and XY respectively.

From this figure and our previous discussion, we now can see that:

1. Under inversion in S_2:

$$A_1 \longleftrightarrow A'_2 \qquad C_1 \longleftrightarrow B_2 \qquad \mathfrak{C}_1 \longleftrightarrow \mathfrak{B}_2$$
$$A_2 \longleftrightarrow A'_1 \qquad C_2 \longleftrightarrow \infty \qquad \mathfrak{C}_2 \longleftrightarrow 0$$
$$A_3 \longleftrightarrow A'_4 \qquad C_3 \longleftrightarrow B_4 \qquad \mathfrak{C}_3 \longleftrightarrow \mathfrak{B}_4$$
$$A_4 \longleftrightarrow A'_3 \qquad C_4 \longleftrightarrow B_3 \qquad \mathfrak{C}_4 \longleftrightarrow \mathfrak{B}_3$$

The spheres J_3, J'_3, J_4, J'_4 remain unaltered while J_2 and J'_2 are interchanged. The spheres S_{12} and S_{34} remain unaltered while $S_{13} \longleftrightarrow S_{24}$, and $S_{14} \longleftrightarrow S_{23}$. The sphere S_2 remains unaltered and so do the isodynamic points while S_3 and S_4 each goes into itself.

2. Under inversion in S_3:

$$A_1 \longleftrightarrow A'_3 \qquad C_1 \longleftrightarrow B_3 \qquad \mathfrak{C}_1 \longleftrightarrow \mathfrak{B}_3$$
$$A_2 \longleftrightarrow A'_4 \qquad C_2 \longleftrightarrow B_4 \qquad \mathfrak{C}_2 \longleftrightarrow \mathfrak{B}_4$$
$$A_3 \longleftrightarrow A'_1 \qquad C_3 \longleftrightarrow \infty \qquad \mathfrak{C}_3 \longleftrightarrow 0$$
$$A_4 \longleftrightarrow A'_2 \qquad C_4 \longleftrightarrow B_2 \qquad \mathfrak{C}_4 \longleftrightarrow \mathfrak{B}_2.$$

The spheres J_2, J'_2, J_4, J'_4 remain unaltered while J_3 and J'_3 are interchanged. The spheres S_{13} and S_{24} remain unaltered while $S_{12} \longleftrightarrow S_{34}$, and $S_{14} \longleftrightarrow S_{23}$. The sphere S_3 remains unaltered as to the isodynamic points while S_2 and S_4 each goes into itself.

3. Under inversion in S_4:

$$A_1 \longleftrightarrow A'_4 \qquad C_1 \longleftrightarrow B_4 \qquad \mathfrak{C}_1 \longleftrightarrow \mathfrak{B}_4$$
$$A_2 \longleftrightarrow A'_3 \qquad C_2 \longleftrightarrow B_3 \qquad \mathfrak{C}_2 \longleftrightarrow \mathfrak{B}_3$$
$$A_3 \longleftrightarrow A'_2 \qquad C_3 \longleftrightarrow B_2 \qquad \mathfrak{C}_3 \longleftrightarrow \mathfrak{B}_2$$
$$A_4 \longleftrightarrow A'_1 \qquad C_4 \longleftrightarrow \infty \qquad \mathfrak{C}_4 \longleftrightarrow 0.$$

The spheres J_2, J'_2, J_3, J'_3 remain unaltered while J_4 and J'_4 are interchanged. The spheres S_{14} and S_{23} remain unaltered while $S_{12} \longleftrightarrow S_{34}$, and $S_{13} \longleftrightarrow S_{24}$. The sphere S_4 remains unaltered as do the isodynamic points while S_2 and S_3 each goes into itself.

The invariance of the configuration under inversions in these four spheres is now established, and its similarity to the configuration associated with the triangle is evident.

Concerning Upper and Lower Bounds of the Roots of a Real Algebraic Equation

By E. C. KENNEDY
Texas College of Arts and Industries

Case I. Consider the real cubic equation

$$f(X)=X^3+a_2X^2+a_1X+a_0=0 \qquad (a_2\neq 0) \tag{1}$$

which may be written as

$$X^3+a_2\left[X+\frac{a_1}{2a_2}\right]^2+a_0-\frac{a_1^2}{4a_2}=0. \tag{2}$$

Since the expression in the bracket is non-negative we can obtain a bound for the roots of (1) by writing

$$X^3\geqslant\frac{a_1^2}{4a_2}-a_0 \qquad (a_2<0) \tag{3}$$

$$X^3\leqslant\frac{a_1^2}{4a_2}-a_0 \qquad (a_2>0) \tag{4}$$

Example I. Find a lower and an upper bound for the real roots of

$$f(X)=X^3-X^2-22X+40=0. \tag{5}$$

By (3), $X^3\geqslant\dfrac{484}{-4}-40=-161$, $X>-5.45$.

To find an upper bound we set $X=Y+b$ obtaining

$$F(Y)=Y^3+(3b-1)Y^2+f'(b)Y+f(b)=0. \tag{6}$$

If we choose $3b>1$ we can apply (4) and get an upper bound. The value of this bound will depend on b and it is desirable to have a scheme for determining a value of b which will give a satisfactory bound.

Writing equation (6) in the form (2), we see that it is desirable that

$$\frac{f'(b)}{f''(b)}+R-b=0, \tag{7}$$

where R is approximately the largest real root of (5).

In our problem we take the largest root of (5) to be 5, say. Then

$$3b^2-2b-22+(5-b)(6b-2)=0,$$

$$b=1\cdot 2 \text{ or } 8\cdot 8.$$

The first value is found to yield the best results. Thus we take $b=1\cdot 2$ or say, 1. This gives $Y^3\leqq 37.1$, $Y<3.34$ whence $X<4.34$.

We note that the ideal value to use for b is $4-\sqrt{6}=1.55$. This gives $X\leqslant 4$, the perfect upper bound. Of course, we would get this value only in the special case where we happened to choose $R=$the largest root of (5).

Summarizing, our formula tells us that

$$-5.45<\text{roots of (5)}<4.34.$$

The perfect bounds are -5 and 4.

The two standard methods given in Dickson's *First Course in the Theory of Equations* give $(-7.33, 23)$ and $(-21, 23)$ as the bounds for (5).

Example II. Find an upper bound for the real roots of

$$X^3+8X^2-9X+C^2=0 \tag{8}$$

$$X^3\leqslant \frac{81}{32}-C^2<2.54-C^2$$

$$X<\sqrt[3]{2.54-C^2}.$$

The two standard methods mentioned above give 4 and 2, respectively, as an upper bound.

If $a_2=0$ or is very small, as in

$$f(X)=X^3+a_1X+a_0=0 \qquad (a_0, a_1 \text{ real}) \tag{9}$$

we set $X=Y+b$, obtaining

$$F(Y)=Y^3+3bY^2+f'(b)\,Y+f(b)=0, \tag{10}$$

after which we apply (3) or (4). Here again b should satisfy

$$\frac{f'(b)}{6b}+R-b=0 \quad \text{or}$$

$$b=\frac{3R\pm\sqrt{9R^2+3a_1}}{3}=R\pm\sqrt{\frac{f'(R)}{3}}, \tag{11}$$

where R is approximately the largest root of (9). Note that $f'(R)$ must be positive or zero.

Example III. Find an upper and a lower bound for the real roots of

$$f(X)=X^3-15X+4=0. \tag{12}$$

Take $R=5$, say. Then (11) gives $b=.53$, say 1. (The other value of b could be used but it is not so satisfactory). Whence

$$Y^3\leqslant\frac{12^2}{12}-(-10)=22,\ Y<2.8,\text{ and } X<3.8,\text{ an upper bound for (12).}$$

If we repeat the process taking $R=3.8$ we get $b=.73$ and this gives $Y<3.0023$ or $X<3.7323$.

To get a lower bound take $R=-3$, say. This gives $b=1$ and $Y>-3.11$ or $X>-4.11$.

To get a better bound take $R=-4.1$, obtaining $b=-.66$, which leads to $X>-4.0008$. Whence

$$-4.0008<\text{roots of (12)}<3.7323.$$

The smallest and largest root of (12) are -4 and $2+\sqrt{3}=3.7321$. This "closing down" process is usually very rapid. Results are obtained quickly and easily if one has at hand ordinary tables.

The above suggests that our scheme might often be used advantageously to approximate the largest and the smallest real roots of a cubic. It is interesting to note that Newton's Method of solving a cubic and our scheme both require the calculation of $f(b)$ and $f'(b)$, but in the latter case there is no division by $f'(b)$. Thus if we choose $R=\sqrt{5}$, a bend point of (12), our method gives $X=4.9$ as a first approximation to the largest root, while Newton's Method gives $X=$ infinity.

Case II. Consider the real quartic

$$f(X)=X^4+a_3X^3+(a_2+1)X^2+a_1X+a_0=0. \tag{13}$$

$$a_3X^3+a_0=-\left(X^2+\frac{a_2}{2}\right)^2-\left(X+\frac{a_1}{2}\right)^2+\frac{a_2^2}{4}\quad\frac{a_1^2}{4}$$

To get a bound for the real roots of this equation we write

$$X^3\geqslant\frac{1}{4a_3}(a_2^2+a_1^2-4a_0),\qquad (a_3<0) \tag{14}$$

$$X^3\leqslant\frac{1}{4a_3}(a_2^2+a_1^2-4a_0),\qquad (a_3>0) \tag{15}$$

Example IV. Find an upper bound for the real roots of

$$f(x)=X^4+25X^3-199X^2-300X+70000=0. \tag{16}$$

By (15), $X^3\leqslant\frac{1}{100}(200^2+300^2-280000)=-1500$, $X<-11.5$.

The bounds obtained by the two standard methods are: $X<18.4$ and $X<12.6$. The least upper bound is about -14.6.

To find a lower bound set $X=Y+b$ obtaining

$$Y^4+\frac{f'''(b)}{6}Y^3+\frac{f''(b)}{2}Y^2+f'(b)\,Y+f(b)=0, \tag{17}$$

and then apply (14) to this equation. Any value of b which satisfies $f'''(b)<0$ or $4b+a_3<0$ will serve to give a lower bound. However, $f'''(b)$ should not be too near zero. In our case we might take $b=-10$, say. As in Case I it is desirable to have a scheme for determining quickly an increasingly accurate value of b, but this appears to be difficult to do in this case. We cannot use this method for approximating arbitrarily closely the largest root as we did in the first case unless we have some means of getting an increasingly accurate value of b.

If $a_3=0$ or is very small we make the same substitution, choosing b so that $f'''(b)<0$ (and not too near zero) to get a lower bound. To obtain an upper bound choose b so that $f'''(b)>0$.

Our scheme can sometimes be used for determining the nature of the roots of an equation of the fourth degree.

Example V: Determine the nature of the roots of

$$X^4+X^3-151X^2-80X+11000=0. \tag{18}$$

By (15) $X^3\leqslant-3624$ or $X<-15.3$. Set $X=Y-1$. We choose $b=-1$ because it is convenient and will serve our purpose—that is, make $4b+1<0$. We have then

$$Y^4-3Y^3-148Y^2+221Y+10929=0 \tag{19}$$

and by (14): $Y^3\geqslant-2277$ or $Y>-13.2$ and $X>-14.2$.

The first result says that *if* the equation has a real root then this root is less than -15.3, while the second part says that no real root can be less than -14.2. Hence (18) has no real roots.

Case III: Our scheme may be partially extended to equations of higher degree. In the case of the quintic it appears to be a little awk-

ward because of the preliminary work required, but for the real sixth degree equation we have

$$f(X)=X^6+a_5X^5+a_4X^4+a_3X^3+(a_2+1)X^2+a_1X+a_0=0 \tag{20}$$

and proceeding as in the cases above we get

$$X^5\geqslant\frac{1}{4a_5}\left(a_3^2+\frac{a_2^2}{a_4}+a_1^2-4a_0\right), \qquad (a_5<0,\ a_4>0) \tag{21}$$

$$X^5\leqslant\frac{1}{4a_5}\left(a_3^2+\frac{a_2^2}{a_4}+a_1^2-4a_0\right), \qquad (a_5>0,\ a_4>0) \tag{22}$$

Note: If $a_4\leqq 0$ in (22) we can change (20) into a satisfactory form by setting $X=Y+b$, where b is chosen so that $f^{(IV)}(b)>0$ and $f^{(V)}(b)>0$. If $a_5=0$ or is very small we proceed in a manner similar to that used in the two preceding cases the second highest power is missing.

Example VI. Find an upper bound for the real roots of

$$X^6+250X^5+100X^4-19999X^2-30,000,000=0. \tag{23}$$

Here $$X^5\leqslant\frac{1}{1000}(4,000,000+120,000,000)=124000,$$

and $$X<10.44.$$

The upper bounds obtained by the two standard methods are 75 and 85500. The largest root is about 10.34.

Sometimes one of the two standard methods will give a better bound than will ours. If one is interested in getting a bound for the roots of a certain equation he would doubtless use all three plans and choose the best result.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

An Appreciation of Sophie Germain

By SISTER M. THOMAS À KEMPIS
College of Saint Teresa, Winona, Minnesota

A search in the *Salle de Manuscrits* in the *Bibliothèque Nationale* for letters that might have been written by either Cauchy or Bolzano disclosed an interesting collection of about fifty letters written by different savants to Sophie Germain. Their transcription and translation revealed Mlle. Germain as a woman of no mean reputation, and immediately aroused special interest in her. An examination of some of the histories of physics showed that she was not even accorded honorable mention. Histories of mathematics, such as Smith's *History of Mathematics*, devote a few sentences to this woman who seems to have occupied a position of honor among the scientists of her day. Mozans, however, in his *Woman in Science* gives a tribute quite worthy of this mathematician.

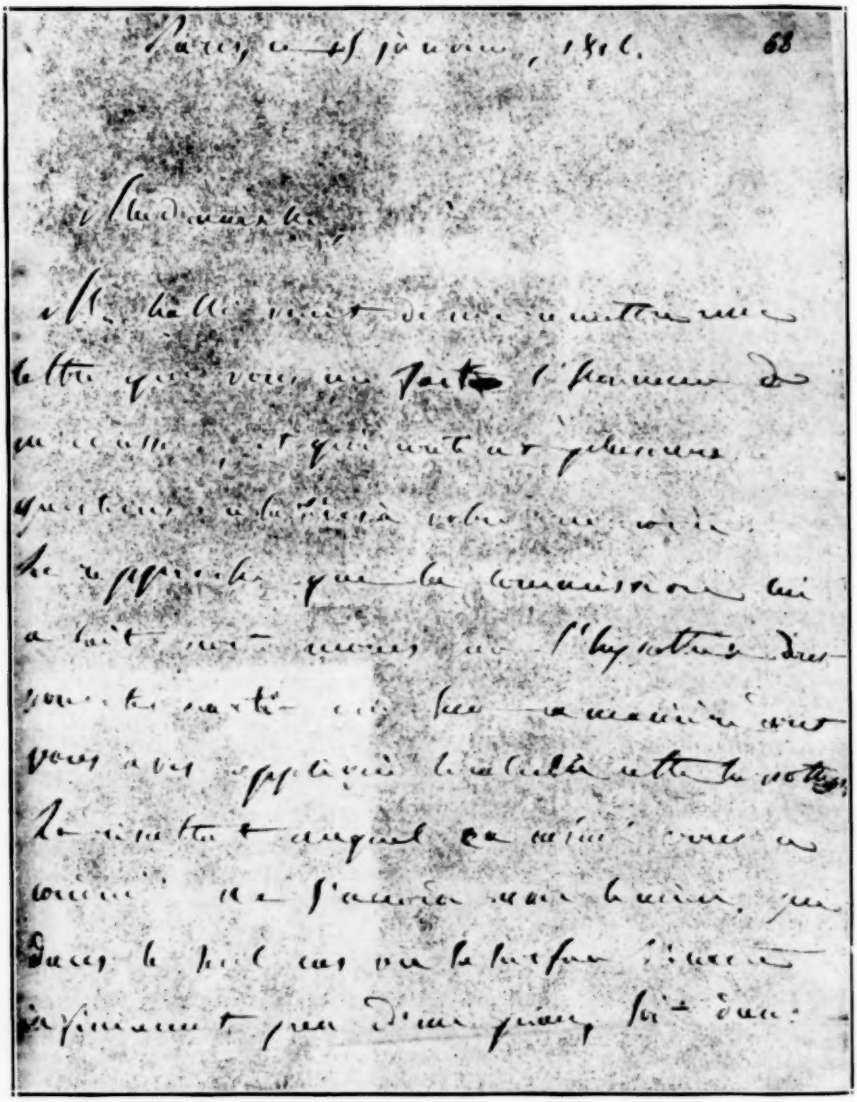
Sophie Germain was born in Paris in 1776 and died there in 1831. This makes her a contemporary of that interesting prodigy, Maria Agnesi. Mozans says:

> Here was the glory of being one of the founders of mathematical physics. A pupil of Lagrange and a co-worker of Biot, Legendre, Poisson, and Lagrange, she has justly been called by De Prony "the Hypatia of the nineteenth century."*

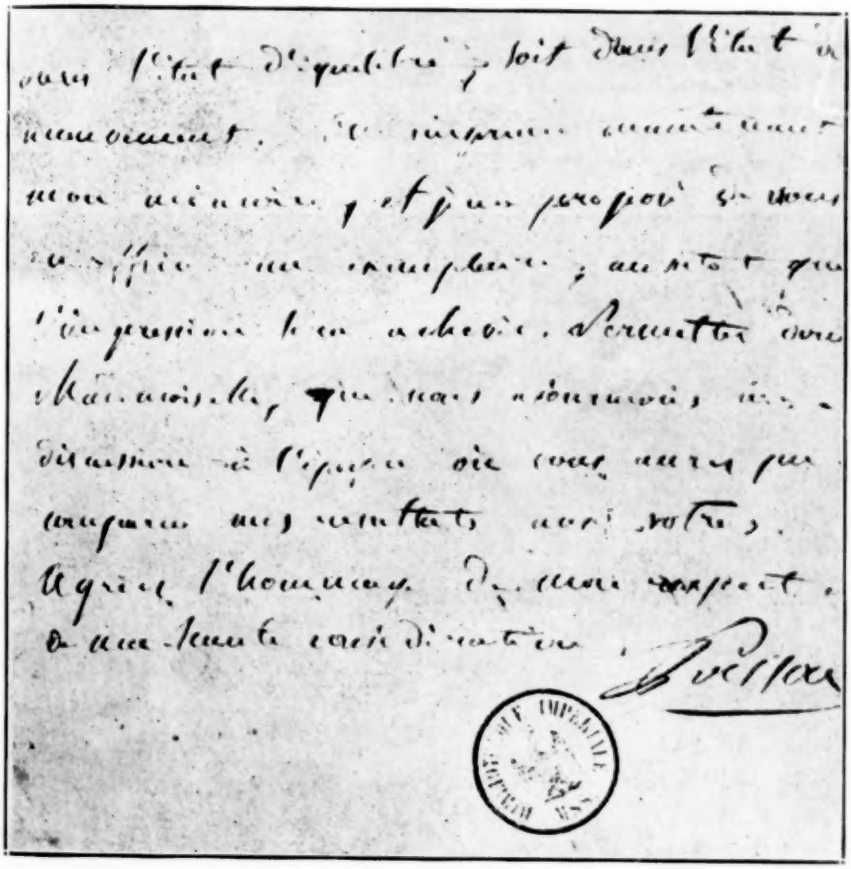
After elaborating on the obstacles that Sophie's family placed in the way of any further study of mathematics, because they felt it out of keeping for a woman, and the strategies she employed to overcome them, Mozans adds:

> Before such determined will, so extraordinary for one of her age, the family of the young Sophie had the wisdom to permit her to dispose of her time and genius according to her own pleasure. And they did well. Like the great geometer of Syracuse, Archimedes, who had ever been her inspiration in the study of mathematics, she would have died rather than abandon a problem which, for the time being, engaged her attention.

*Mozans, *Woman in Science*, 1913, p. 154.



A Communication of Poisson to Mlle. Germain, Paris, 1816.



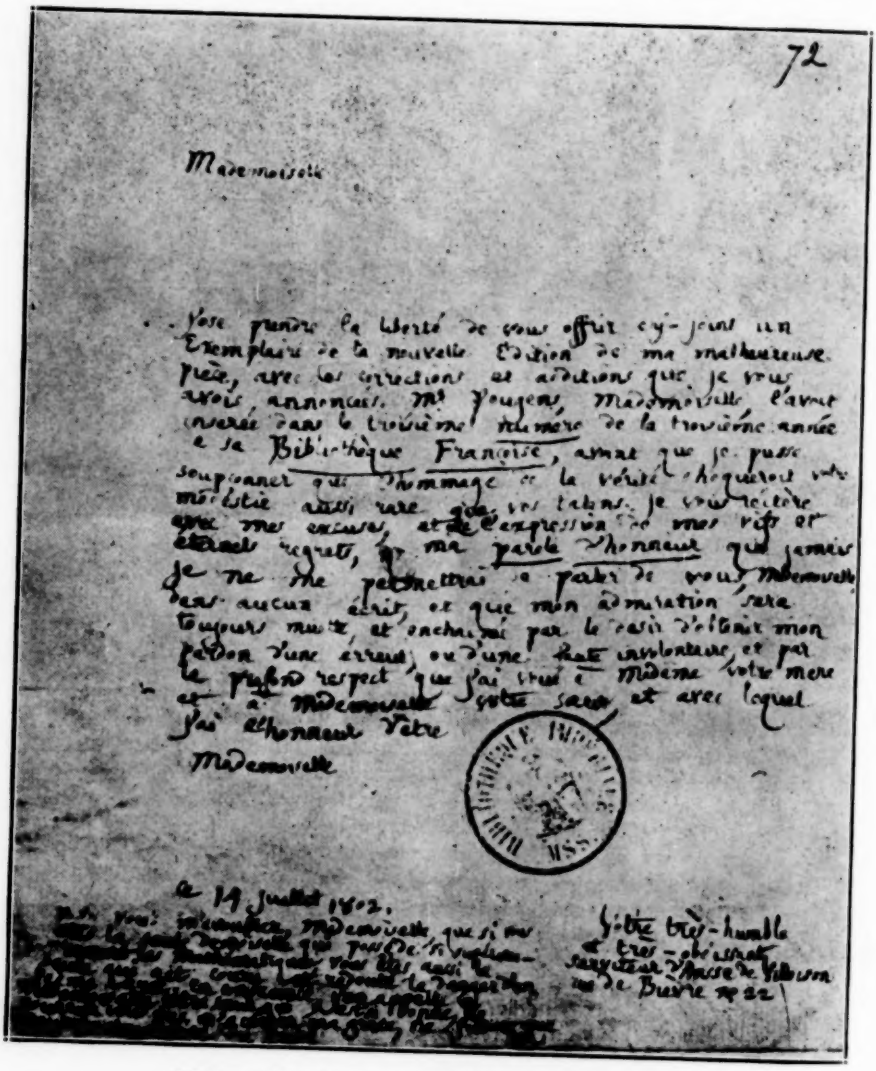
She first attracted the attention of savants by her mathematical theory of Chladni's figures. By the order of Napoleon, the Academy of Science had offered a prize to the one who would "Give the theory of the vibration of elastic surfaces and compare it with results of experiment." Lagrange declared the problem insoluble without a new system of analysis, which was yet to be invented. The consequence was that no one attempted the solution except one who, until then, was almost unknown in the mathematical world; and this one was Sophie Germain.

Great was the surprise of the savants of Europe when they learned that the winner of the *grand prix* of the Academy was a woman. She became at once the recipient of congratulations from the most noted mathematicians of the world. This evidently brought her into scientific relations with such eminent men as Delambre, Fourier, Cauchy, Ampère, Navier, Gauss and others already mentioned.

It was in 1816, after eight years of work on the problem that her last memoir on vibrating surfaces was crowned in a public séance of the *Institut*

de France. After this event Mlle. Germain was treated as an equal by the great mathematicians of France. She shared their labors and was invited to attend sessions of the *Institut*, which was the highest honor that this body had ever conferred on a woman.*

It is not so much to elaborate on her life that these facts are added to those already written of her. It is rather to strengthen Mozan's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Mademoiselle

Vous prendre la liberté de vous offrir ci-joint un Exemplaire de la nouvelle Edition de ma malheureuse pièce, avec les corrections et additions que je vous avois annoncées. Mr Fourcroy, Mademoiselle, l'avait insérée dans le troisième volume de la troisième année de sa Bibliothèque Française, ainsi que je puis soupçonner que l'hommage de la vérité chaqueroit votre modestie aussi rare que vos talens. Je vous réitère avec mes excuses, et de l'expression de mes vifs et éternels regrets, et ma parole d'honneur que jamais je ne me permettrai de parler de vous Mademoiselle dans aucun écrit, et que mon admiration sera toujours muette, et enchaînée par le désir d'obtenir mon pardon d'une erreur, ou d'une faute involontaire, et par le plus respectueux que j'ai voué à Madame votre mère et à Mademoiselle votre sœur et avec lequel j'ai l'honneur d'être

Mademoiselle

Votre très-humble et très-obéissant serviteur Vallisson rue de Bievre n° 22

le 14 Juillet 1802.

A letter of Villoisson to Sophie Germain, July 14, 1802.

*Mozans, *Woman in Science*, 1913, pp. 154-5.

conclusions that "all things considered, she was the most profoundly intellectual woman France has produced."* This is substantiated by a study of a few letters of the men of genius previously mentioned. Though their recital may prove somewhat monotonous, nevertheless it is just such source material that will awaken a proper appreciation of her true worth.

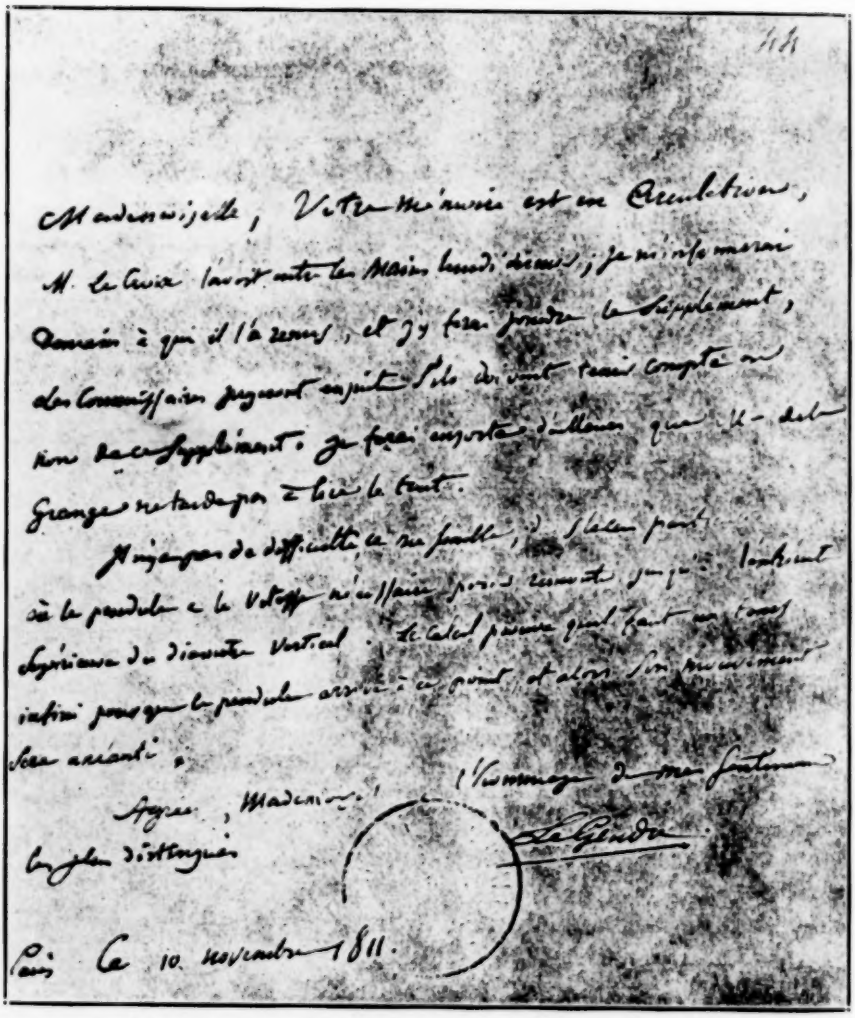
A letter of Villoisson, a Hellenist, in fact a professor of Greek, is amusing. He referred to her in a Latin poem, which is preserved as an abstract in the Sophie Germain collectio nin the *Bibliothèque Nationale.* It is, no doubt, a reprint from the review called, *Bibliothèque Française.* The poem is too long to include here; but, judging from Villoisson's letter, she was not too pleased that he attempted to bring to light her hidden talents. She must have chided him, because the letter dated July 14, 1802 contains the following apologies:

> MADEMOISELLE:
>
> I dare to take the liberty to offer you, adjoined to this, a sample of the new edition of my unfortunate work with corrections and additions which I have told you about. M. Paugens, Mademoiselle, had inserted it in the third issue of his *Bibliothèque Française* before I became suspicious that the homage of the truth would shock your modesty, which is as rare as your talent. I repeat with my excuses and lively and eternal regret on my word of honor that I should never have permitted myself to speak of you, Mademoiselle, in writing, and that my admiration will always be silent and enchained by the desire to obtain pardon for an error, or for an involuntary fault, and by the deep respect which I have vowed to your mother and sister. I have the honor to be, Mademoiselle,
>
> Your humble and obedient servant,
>
> ANASE DE VILLOISSON,
>
> Rue de Biever, No. 22.

Sophie Germain evidently freely submitted the results of her research to Le Gendre. He, in turn, carefully read her works and in order to do them justice he frequently requested some of his colleagues, perhaps better versed in certain phases of the science than he was, to make a careful correction of all her data. This statement is corroborated by many such letters as the following:

> MADEMOISELLE:
>
> Your memoir is in circulation. M. de la Croix took it last Monday. I am informed that he will send it back tomorrow and I shall add a supplement to it. The commissioners can finally judge whether they ought to take account of this supplement or not. I shall see to it that M. de la Grange does not put off reading the entire work.
>
> There is no difficulty, it seems to me, in the particular case where the pendulum has the necessary speed to ascend nearly to the highest point of a

Ibid., p. 156.



A letter of Le Gendre to Sophie Germain. Paris, November 10, 1811.

vertical diameter. This calculation proves that an infinite amount of time is necessary until the pendulum arrives at this point and then its movement will be destroyed.

Accept, Mademoiselle, the token of my most distinguished sentiments.

LE GENDRE.

Paris, November 10, 1811.

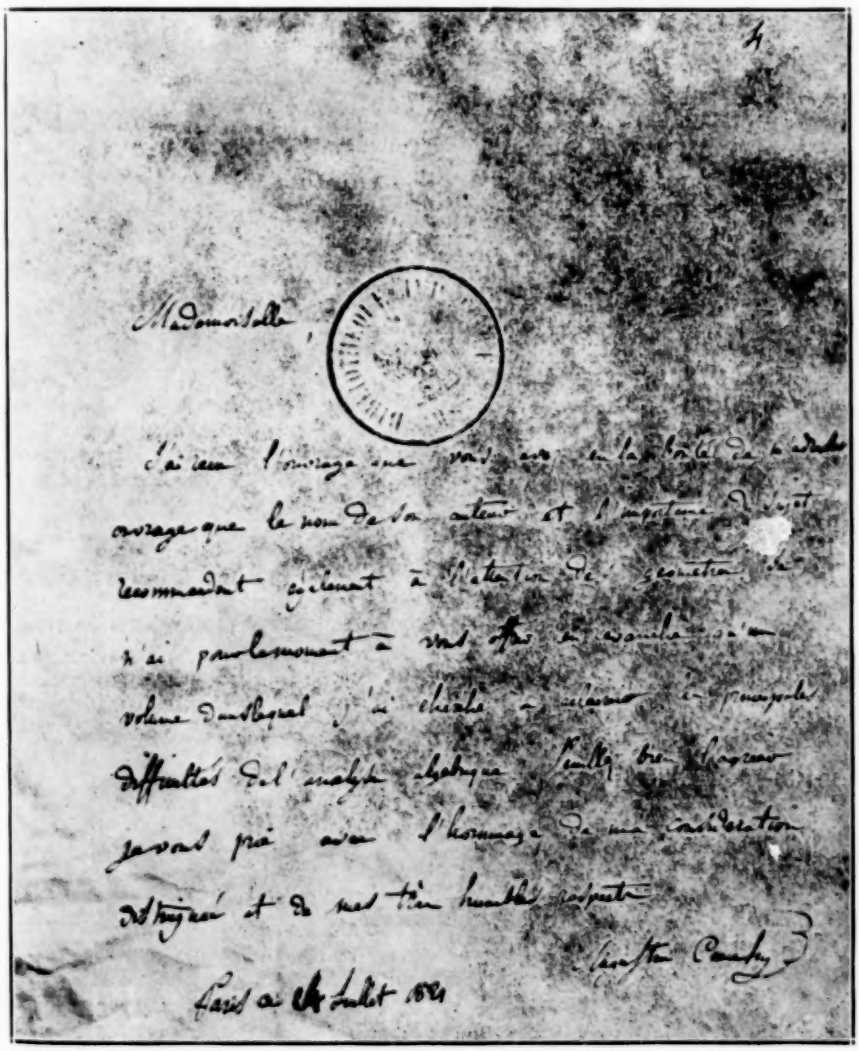
On December sixth of the same year, Le Gendre gave the final report on her work. He was, like many directors in research, very frank, but withal very gentle and ever anxious to shield.

MADEMOISELLE:

I have no good news to give you concerning the examination of your memoir. They find that your principal equation is not exact. ...

He, then, gives the findings of Lagrange on the same point, as well as Euler's conclusions. He continues (and observe the encouragement):

... The source of your error appears to be in the memoir in which you believed you were able to deduce the equation of a surface from the equation of a simple wave. They do not lend themselves at all to the substitutions



A letter from Cauchy to Mlle. Germain, Paris, July, 1821.

which you have used. The equation for a surface must follow the method indicated by La Grange in his 11th edition, p. 118, in which a suitable term is added to represent force endowed with elasticity. As to the rest, things are subject to particular difficulties which have not yet been developed and there would be objections to make against the analysis of the very article I have cited.

I cannot render less justice to these efforts which are praiseworthy in themselves, although they do not have the effect which I have desired. But this is even more of a reason to preserve your incognito and on my part I promise to keep the deepest silence.

Accept, Mademoiselle, the affections of your devoted servant,

LE GENDRE.

December 6, 1811.

One can readily infer the influence Sophie had on analysis of vibrating surfaces. Her very failures became triumphs, because she stimulated investigations in this field by such profound scholars as Lagrange, Laplace and Biot.

The esteem in which she was held by Cauchy is expressed in a simple note. His notes could be nothing else, since Cauchy was capable of that simplicity which is so characteristic of truly great souls. He writes:

MADEMOISELLE:

I have the work which you had the kindness to send me—a work, the name of whose author and the importance of whose subject recommended it equally to the attention of geometers. At this time I can only offer you in return a volume in which I have sought to clarify the difficult principles of algebraic analysis. Please accept it, I pray you, together with the testimony of my deep consideration and my very humble respects.

AUGUSTIN CAUCHY.

Paris, July 8, 1821.

The work that she sent him was, perhaps, *Un memoire sur les vibrations des lames elastiques* which was published in 1820. She had received special recognition for this very memoir from the Institute of France in 1815. On July 23 of the same year, Delambre, as the Life-Secretary of the Academy of Sciences, in the name of the Institute of France wrote as follows:

MADEMOISELLE:

The Academy has received with the greatest pleasure the work that you wished very much to send to it and which is entitled: *Researches on the Theory of Elastic Surfaces*, which you have just published. The Academy has charged me to thank you in its name for the work of this interesting memoir which has been honorably placed in the library of the Institute and also to express the Academy's recognition of this new proof that you have given her of your talents.

Accept, I beg of you, Mademoiselle, the homage of my respect,

DELAMBRE.

Institut de France

Académie Royale des Sciences

Paris, le 3 juillet 1821.

Le Secrétaire perpétuel de l'Académie.
à Mademoiselle Sophie Germain

Mademoiselle

L'Académie a reçu avec le plus grand intérêt l'ouvrage que vous avez bien voulu lui adresser et qui est intitulé: Recherches sur la théorie des surfaces élastiques que vous venez de publier. Elle me charge de vous en remercier, en son nom, de l'envoi de cet ouvrage intéressant qu'elle a fait déposer honorablement dans la bibliothèque de l'Institut, et de vous exprimer sa reconnaissance de cette nouvelle preuve que vous lui donnez de votre talent.

Agréez je vous prie, Mademoiselle, l'hommage de ma considération.

Delambre

Mademoiselle Sophie Germain

In 1805, Fourier writes Mlle. Germain making an appointment with her to go over a memoir of hers submitted to him by Le Gendre. Then in 1826 he writes in the name of the Institute:

> MADEMOISELLE:
>
> The Academy has received the work which you wished to send to it and which is entitled, ***Remarks on the Nature, the Limits and the Extent of the Question of Elastic Surfaces and the General Equations of these Surfaces.***
>
> I have the honor of thanking you in the name of the Academy for your sending of this work. M. Cauchy has been appointed to make a verbal report on it. This volume will be placed in the library of the Institute.
>
> I have the honor to offer to you the assurance of my respect.
>
> FOURIER.

The space is too limited to quote letters of equally great import. It becomes apparent that Sophie Germain exerted no small influence on the science and mathematics of her time. Mozans comments:

> And yet, strange as it may seem, when the state official came to make out the death certificate of this eminent associate and co-worker of the most illustrious members of the French Academy of Science, he designated her as a *rentière-annuitant*—not as a *mathématicienne*. Nor is this all. When the Eiffel Tower was erected in which the engineers were obliged to give special attention to the elasticity of the materials used, there were inscribed on this lofty structure the names of seventy-two savants. But one will not find in this list the name of that daughter of genius, whose researches contributed so much toward establishing the theory of the elasticity of metals, Sophie Germain. Was she excluded from this list for the same reason that Agnesi was ineligible in the French Academy—because she was a woman? It would seem so. If such, indeed, was the case, more is the shame for those who were responsible for such ingratitude toward one who had deserved so well of science, and who by her achievements had won the enviable place in the hall of fame.*

*Mozans, *Woman in Science*, 1913, p. 156.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Why Logarithms to the Base *e* Can Justly be Called Natural Logarithms

By JOHN ELLIS EVANS
The Ohio State University

In searching through the literature I found essentially only three reasons why logarithms to the base e can justly be called natural logarithms. The three reasons are: (1) e is a quantity which arises frequently and unavoidably in nature, (2) natural logarithms have the simplest derivatives of all the systems of logarithms, and (3) in the calculation of logarithms to any base, logarithms to the base e are first calculated, then multiplied by a constant which depends on the system being calculated.

A. *How e Arises in Nature.* Probably the most obvious reason for calling logarithms to the base e natural logarithms is that e is a number which very often puts in its appearance in problems relating to many different types of natural phenomena. Thus the law of natural growth illustrates how the number e is unavoidable in nature.

Let us consider the problem of bacteria increase. Let N_0 be the number of bacteria in a culture at time t_0 and N_t the number in the culture after t hours. If the physical conditions of the culture are kept constant, the bacteria will increase at a certain rate r. Now we can write the following expressions:

$$N_1 = N_0\left(1+\frac{r}{100}\right)^1$$

$$N_2 = N_0\left(1+\frac{r}{100}\right)^2$$

. . . . . .

$$N_t = N_0\left(1+\frac{r}{100}\right)^t$$

Now these expressions are only approximations, for some of the new bacteria may reach the age of reproduction before the end of the hour. A better approximation can be obtained by considering shorter time intervals. Now by replacing

$$\frac{r}{100} \text{ by } R$$

and by considering half-hour intervals, where t is in hours, we can rewrite the last expression above as follows:

$$N_t = N_0\left(1+\frac{R}{2}\right)^{2t}$$

For an accurate expression for the phenomenon we will have to consider the limit of the above expression as the time intervals approach zero.

$$N_t = \lim_{n\to\infty}\left(1+\frac{R}{n}\right)^{nt} = N_0 e^{Rt} *$$

where e is defined by the infinite series

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots$$

This same reasoning may be applied to such natural phenomena as: the expansion or contraction of heated objects, the decay of radioactive material, the amount of a substance in a solution at any given time where there is an inexhaustible supply of the solute, the amount of a given material present at any time in a given chemical reaction, the barometric pressure at any altitude, retardation of falling bodies due to air resistance, the absorption of light in passing through any thickness of a given transparent material, damped vibratory motion, and natural growth. Lord Kelvin refers to such phenomena as instances of the "Compound Interest Law in Nature." This law is also known as the "Snow Ball Law", or the "Law of Natural Growth."

There are quite a few quotations to be found in the literature supporting this reason for calling logarithms to the base e natural logarithms. W. M. Feldman† writes, "The thoughtful student, who has most probably wondered why such a peculiar base was chosen, will now have found the solution to the riddle. The number 2.71828.. is

*For the proof of this expression, see pp. 19-21 of John Ellis Evans' *A Critical Study of Natural Logarithms*, a thesis presented for the degree of Master of Arts, at the Ohio State University, 1937.

†*Biomathematics* (London: Charles Griffin & Company, Ltd., 1923) p. 80.

merely the sum of the series e which is not only so very useful for the purpose of calculating logarithms, but occurs in the consideration of all natural phenomena which take place in accordance with the compound interest law", and Hitchcock and Robinson say, "One reason why this base [e] is properly called the 'natural' base is that it enters inevitably and naturally in so many problems of physics and chemistry." Similar statements have been made by Bisacre[1]; Cowles and Thompson;[2] Griffin;[3] Helliwell,[4] Tilley, and Wahlert; Koch,[5] and Shibli.[6]

B. *Natural Logarithms Have Simpler Derivatives Than Other Systems of Logarithms.* Another reason why logarithms to the base e can justly be called natural logarithms is that this system has the simplest derivative of all the systems of logarithms.

$$y = \log_e x$$

$$\frac{dy}{dx} = \frac{1}{x}.$$

Now we will consider the derivative of the logarithmic function having a general base b.

If $$y = \log_b x = \frac{\log_e x}{\log_e b}$$

then $$\frac{dy}{dx} = \frac{\log_e b \cdot 1/x}{(\log_e b)^2} = \frac{1}{\log_e b} \cdot \frac{1}{x}$$

and it is obvious that this constant factor

$$\frac{1}{\log_e b}$$

appears in all the successive derivatives of $\log_b x$. The simplification resulting from the use of $\log_e x$ instead of $\log_b x$ is very important in theoretical considerations where differentiation is involved.

[1] *Applied Calculus* (London: Blackie and Son, Ltd., 1922) p. 245.
[2] *A Textbook of Trigonometry* (New York: D. VanNostrand Co., Inc., 1936) p. 322.
[3] *An Introduction to Mathematical Analysis* (Revised Edition, New York: Houghton Mifflin Co., 1936) p. 253.
[4] *Fundamentals of College Mathematics* (New York: MacMillan Company, 1935) p. 334.
[5] *The Mathematics of Applied Electricity* (New York: John Wiley and Sons, Inc., 1912) p. 102.
[6] *Plane and Spherical Trigonometry with Applications* (New Edition, New York: Ginn & Company, 1936) pp. 32-33.

This idea is expressed in the statements of Bisacre;[1] Griffin;[7] Knopp;[8] Lasley and Browne;[9] Ch. J. de La Vallée Poussin;[10] Schlömilch;[11] and Wilczynski.[12] Griffin's quotation, which follows, is typical of the group. "Observe that e is the natural base to use in problems requiring the differentiation of the logarithm—because of the simplicity of the formula

$$\frac{dy}{dx}=\frac{1}{x} \text{ as compared with } \frac{dy}{dx}=\frac{M}{x}.$$"

In the above quotation M is used to represent the constant

$$\frac{1}{\log_e b}$$

and is known as the modulus of the system of logarithms having b as a base. Another quotation from the article on logarithms in the Encyclopædia Britannica is as follows: "The base 10 yields logarithms that are most convenient for the purposes of computation; the base 2.71828... yields logarithms which lead to simpler formulas in higher analysis than other systems and are therefore the most 'natural' ones to use." This quotation tends to emphasize the naturalness of logarithms to the base e and the artificiality of logarithms to the base 10. Logarithms to the base 10 are used as a device to simplify numerical computation while natural logarithms are used because they simplify the analytic expression of natural phenomena.

C. *Natural Logarithms are Fundamental in the Calculation of Logarithms to Any Base.* Probably the most important reason for calling logarithms to the base e natural logarithms is that in calculating the logarithms of any system they are first calculated to the base e and then multiplied by a constant called the modulus of the system being calculated.

In order to calculate logarithms we expand $\log_b x$ by Maclaurin's or Taylor's expansions. These expansions require the successive de-

[7] *An Introduction to Mathematical Analysis* (Revised Edition, New York: Houghton Mifflin Co., 1936) p. 238.

[8] *Theory and Application of Infinite Series.* Translation from the Second German Edition by Miss R. C. Young (London: Blackie and Son, Limited, 1928) p. 56.

[9] *Introductory Mathematics* (New York: McGraw Hill Book Company, Inc., 1933) p. 219.

[10] *Cours d'Analyse Infinitesimale,* (Sixieme Edition, Paris: Gauthier-Villars, 1926) Tome I, p. 35.

[11] *Compendium der Höheren Analysis,* (Braunschweig: Friedrich Vieweg und Sohn, 1881), Band I, Seite 25.

[12] *College Algebra with Applications* (New York: Allyn and Bacon, 1916) pp. 295-296

rivatives of $\log_b x$. Now the successive derivatives of $\log_b x$ are the successive derivatives of $\log_e x$ times a constant factor

$$\frac{1}{\log_e b}.$$

Since $$\frac{1}{\log_e b}$$

occurs in each term of the expansion, we factor it out of the expansion, giving us

$$\log_b x = \frac{1}{\log_e b} \text{ (the expansion of } \log_e x)$$

Therefore we first calculate logarithms to the base e then multiply them by the modulus of the system desired. Thus natural logarithms seem to be a basic system while other systems are but variations of natural logarithms.

Graham[13] and John, Feldman,[14] and Williamson[15] have made statements supporting this reason for calling logarithms to the base e natural logarithms. The statement of Williamson is as follows: "Also, since the calculation of logarithms to any other base starts from the logarithms of some numbers to the base e; and moreover, since the logarithms of all numbers are expressed by their logarithms to the base e multiplied by the modulus of transformation, the system whose base is e is fundamental in analysis, . . . ".

[13] *Advanced Algebra* (Revised Edition, New York: Prentice-Hall, Inc., 1936) p. 35.
[14] *Biomathematics* (London: Charles Griffin & Company, Ltd., 1923) p. 79.
[15] *An Elementary Treatise on the Differential Calculus* (Sixth Edition, London: Longmans, Green, and Company, 1887) pp. 25-26.

More About Better Mathematics

By Joseph Seidlin

The substance and musings of this article are suggested and implemented by a recent study by Lee Emerson Boyer.* The "primary purpose of the study is *aimed* at the needs (in mathematics) of prospective teachers". The "needs" may be satisfied by "valid subject matter in a *general* mathematics course." It is outside the scope of this article to review the study or to criticize the techniques employed, or to question the validity or the relevancy of the "evidence" submitted. Our readers may choose to evaluate the study without any outside interference. Mr Boyer might welcome any criticism and perhaps even supply the inquirers with copies of the study at a nominal cost. (Our advertising department may justifiably object to the preceding statement. It *is* free publicity.)

To return to Mr. Boyer's study: There seems to be not the slightest doubt about the desirability of a general course in mathematics for secondary school teachers. While, naturally "experts" disagree on the nature or the emphasis of the units of content, the study "reveals" several factors with respect to such a course:

"It is preferable to cover a few units thoroughly than many units superficially. As pointed out . . . there is considerable danger that the course may become one of light reading and shallow discussion instead of a scholarly study of well chosen exercises that will challenge the student to use judiciously the thought material gained from the reading and discussion."

Those who argue against a general course claim that even "fewer units" still cover too much ground for anything but superficial treatment.

"The success or failure of a course in college general mathematics does not depend wholly upon well chosen course materials but also upon an able teacher, good teaching methods, and conscious effort to develop desirable habits of neatness, order, perseverance, suspended judgment, 'straight thinking', and the like."

It seems that we cannot get away from teaching. It may be that after all these years of building courses we may discover—as some of

**College General Mathematics for Prospective Secondary School Teachers*, Lee Emerson Boyer, The Pennsylvania State College, Studies in Education, Number 17.

us suspect—that it would have been more productive of results to put all that energy and genius into "building" teachers.

There are other interesting and worth-while findings and recommendations. Now and then one finds an exaggeration out of place in so careful a study. For instance, on page 98: "Thus there is a widespread demand in teacher training circles that prospective secondary school teachers receive a *broad* and *liberal* education which will acquaint them with all the major fields of human learning." What does "acquaint" mean? If a broad and liberal education is to include *all* the major fields of human learning then of necessity the acquaintance with them must be the mere nodding kind. And is not that kind of little knowledge very, very dangerous? Is not one of our present major intellectual afflictions too much spread of too little knowledge?

How much mathematics must one learn "to acquire a new sense of mathematical power, insight, and confidence"? Much more than even that provided by an ideally constructed two-semester course in general mathematics.

Is it not time for us to quit harping on the theme that our present courses in mathematics—algebra, geometry, trigonometry, analytic geometry, and calculus—*with emphasis on the subject's formal and narrowly technical aspects* care only for the prospective specialists in the mathematically technical sciences? If our present courses are as poor as they are rumored to be then they are *not* capable of producing prospective specialists and they certainly do not *care* for *any* type of student.

One wonders, too, whether any good teacher of algebra, geometry, etc., is guilty of the following charge: "Almost no attempt is made to show the subject's social history, its significance in our social lives, and the immense dependence of mankind upon it." On the other hand a poor teacher of general mathematics may stress these extra subject values to the exclusion of the subject itself.

It is not so much the treatment of the topic or the findings or the conclusions of Mr. Boyer, as the topic itself, in general, which prompts the following outpouring.

In the teaching of mathematics as in the teaching of every other subject, much that is written has a poetic tint and much is expressed in such fine prose as to make it literature of high order with all the privileges and immunities and license appertaining to the products of men of letters. Again, much that is written is loosely idealized and wishful philosophy (in the non-technical sense). As for specialists in mathematics, they are like any other set of specialists. They are, as they should be, and as they cannot help being, biased. Principally

this prejudice is of two kinds—the first: that mathematics is the subject of angels and only those fit to live on holy ground should be permitted to play with it; the second: that mathematics both in its development and in the part it could play in the development of the human race is a panacea for all our ills. The latter group would permeate all our school curricula with courses in mathematics so that every man, woman, and child would reason clearly and scientifically and would understand completely the ever increasing complexity of our environment.

We need to consider more carefully the composition of the second group. Almost like a political organization it has its right wing, left wing, and center. There is little disagreement in the whole group with regard to the possible good that mankind would get from the growth and study of mathematics. There is much disagreement as to the ways and means and materials in mathematics which will be of greatest benefit to our suffering human race.

The right wingers insist on the disciplinary value of the subject and claim that there is no general mathematics:—that, if a person is to understand his mathematical environment he is to gain that understanding through hard mental labor in the acquisition of facts and principles in the several branches of mathematics. It follows almost as a corollary that the study of mathematics is not for all; that much of our population must be content with rudiments and manipulative schemes in the use of everyday, mathematical necessities.

The left wingers claim that the study of mathematics, like the study of every other branch of human learning, must be considered chiefly as play; that the branches of the subject are merely accidental, traditional development; that, therefore, general mathematics is the thing; that in the everyday workings of mathematics there is no separation of a problem into algebra, geometry, analytic geometry, calculus or trigonometry; that it is all "just mathematics"; that if children are inspired so that they may love mathematics, and all children are potentially capable of loving mathematics, they will grow up into men and women not only better able to understand their complex environment but more likely to give direction to their destiny. It is easy to see how either group would invent and propagate all sorts of slogans, and incisively stated gems of philosophy, which might serve the purpose of increasing the membership of that particular group. Unfortunately for the cause of the growth and development of mankind these slogans and gems of philosophy seem so irresistibly convincing that even mathematicians whose life work it is to scrutinize carefully both the premises and consistency of reasoning somehow succumb to the enticement.

The centrists are comparatively few in number and, like centrists in other fields, occasionally shift to the left or to the right. It is therefore difficult to describe them or to be able to state their position at any time. Roughly they believe that mathematics is an important branch of human learning; that it is fundamental to the understanding and the workings of other branches of human learning; that it has quite unique and important values in the educative process; that in its higher branches it requires if not special then general intelligence of high order; that in the application of its fundamentals it is of interest and within the power of comprehension of the greater number of normal people. They believe, too, that some of the types of reasoning characterizing the subject of mathematics would with some alteration or modification be applicable to fields other than mathematics; that, therefore, such human ills and sufferings and discomforts as may be traced to lack of understanding or reasoning or analysis or scientific detachment could be reduced somewhat if not wholly eliminated by a more general inoculation of our citizenry with mathematical tools and principles. On the other hand they are keenly, albeit unhappily, aware of the fact that the greater part of human ills and sufferings and discomforts is not caused by lack of understanding, reasoning, analysis or scientific detachment and therefore cannot be reduced or eliminated, even if it were possible to make a high grade mathematician out of every man, woman and child in our society. In other words, social progress depends on many, many factors that even an ideal study of ideal subject matter in mathematics does not involve.

We do not put the freshman intelligence through any worthwhile educational mills when we constantly dose him mathematical materials previously reduced to powder or tablet form by lecture process. The physical digestive machinery requires steady exercise if its powers are to be maintained at any high level, and this means that it must be furnished with the sorts of food that yield the exercise. Similarly, the power of intellectual digestion (mathematical power?) does not grow, but only shrinks under a regimen of pre-digested mental foods. Let the instructor pulverize all the lumps and knots in mathematics and the outcome can only be a flabby-minded pupil. He will be without mental initiative, without sustained concentrative ability—without either constructive or analytical power.—From *Mathematics News Letter*, vol. VI, number 4.

The Build Up of Current In An Iron Clad Circuit

By Lynn L. Merrill
Rensselær Polytechnic Institute

Although the following problem is one of electrical engineering it is almost entirely mathematical and should be of interest to mathematicians.

The differential equation for the decay of current in an iron clad circuit as given by Steinmetz in his text *Transient Phenomena* is

$$(1) \qquad \frac{na10^{-8}}{(1+bi)^2}\frac{di}{dt}+nc10^{-8}\frac{di}{dt}+Ri=0$$

where i = current at any time t and n, a, b, c and R are constants of the circuit.

For simplicity the equation (1) may be rewritten as

$$(2) \qquad \frac{M}{(1+bi)^2}\frac{di}{dt}+N\frac{di}{dt}+Ri=0.$$

where M replaces $na10^{-8}$ and N replaces $nc10^{-8}$.

The solution, which is given by Steinmetz, may be obtained easily by separation of variables and is

$$(3) \qquad t-t_0=\frac{M}{R}\left(\frac{1}{1+bi_0}-\frac{1}{1+bi}\right)+\frac{N}{R}\log\frac{i_0}{i}+\frac{M}{R}\log\frac{i_0(1+bi)}{i(1+bi_0)},$$

where i_0 = current at time t_0.

The differential equation for the build up of current with a constant voltage E applied to the circuit is not given by Steinmetz but may be obtained by rewriting equation (2) with right hand member E. Thus,

$$(4) \qquad \frac{M}{(1+bi)^2}\frac{di}{dt}+N\frac{di}{dt}+Ri=E.$$

Equation (4) cannot be solved directly but may be transformed into an equation in which the variables are separable by rewriting it as

$$\frac{M}{(1+bi)^2}\frac{di}{dt}+N\frac{di}{dt}+R\left(i-\frac{E}{R}\right)=0 \tag{5}$$

and replacing

$$\left(i-\frac{E}{R}\right) \text{ by } I \text{ and } \frac{di}{dt} \text{ by } \frac{dI}{dt},$$

$$\frac{M}{\left(1+\frac{bE}{R}+bI\right)^2}\frac{dI}{dt}+N\frac{dI}{dt}+RI=0, \tag{6}$$

or, taking

$$1+\frac{bE}{R}=k,$$

$$\frac{M}{(k+bI)^2}\frac{dI}{dt}+N\frac{dI}{dt}+RI=0. \tag{7}$$

Separating the variables in equation (7)

$$\frac{M+N(k+bI)^2}{I(k+bI)^2}dI+Rdt=0. \tag{8}$$

Then, by the use of partial fractions,

$$\frac{M+N(k+bI)^2}{I(k+bI)^2}\equiv\frac{A}{I}+\frac{B}{(k+bI)^2}+\frac{C}{k+bI}$$

and, taking I successively as, $0, -\frac{k}{b}$ and 1,

$$A=\frac{M+NK^2}{K^2}$$

$$B=C=-\frac{Mb}{K}.$$

Using these values and integrating, the solution of equation (7) becomes

$$(9)\qquad T=\frac{1}{R}\left[-\frac{M}{K}\quad\frac{1}{K+bI}-\frac{M+NK^2}{K^2}\log I+\frac{M}{K^2}\log(K+bI)\right]+C_1,$$

C_1 being the constant of integration. Replacing I by

$$\left(i-\frac{E}{R}\right) \text{ and } K \text{ by } \left(1+\frac{bE}{R}\right)$$

in equation (9) the solution of (4) becomes

$$(10)\qquad T=\frac{1}{R}\left[\frac{M}{\left(1+\frac{bE}{R}\right)^2}\log(1+bi)-\frac{M}{1+\frac{bE}{R}}\quad\frac{1}{1+bi}-\frac{M+N\left(1+\frac{bE}{R}\right)^2}{\left(1+\frac{bE}{R}\right)^2}\log\left(i-\frac{E}{R}\right)\right]+C_1$$

or, determining C_1 from the fact that $i=i_0$ when $t=t_0$,

$$(11)\qquad T-T_0=\frac{1}{R}\left[\frac{M}{\left(1+\frac{bE}{R}\right)^2}\log\frac{1+bi}{1+bi_0}+\frac{M}{1+\frac{bE}{R}}\left(\frac{1}{1+bi_0}-\frac{1}{1+bi}\right)+\frac{M+N\left(1+\frac{bE}{R}\right)^2}{\left(1+\frac{bE}{R}\right)^2}\quad\log\frac{i_0-\frac{E}{R}}{1-\frac{E}{R}}\right]$$

Now, when the circuit constants are known equation (11) may be used to plot current against time. The curve so obtained is found to be in very close agreement with that obtained with an oscillograph if care is taken to demagnetize the coil before recording the current.

Mathematical World News

Edited by
L. J. ADAMS

Applications are being received for Benjamin Peirce Instructorships in Mathematics at Harvard University for the academic year 1940-41. Candidates should ordinarily have the doctorate or its equivalent. Applications should be sent to, and further information may be received from, the Chairman of the Department of Mathematics.

Louisiana Polytechnic Institute announces the appointment of Dr. H. S. Gatewood of the University of Wisconsin to an assistant-professorship in its department of mathematics.

Dr. Allen A. Shaw, associate professor of mathematics in the University of Arizona, died on October 11, 1939. Dr. Shaw was well known in mathematical circles on the Pacific coast, and contributed several articles to this magazine.

The mathematics and engineering section of the Southern California Junior College Conference was addressed by Dr. Philip Biegler, Dean of the College of Engineering of the University of Southern California on October 21, 1939. His remarks were concerned with the guidance of pre-engineering students.

The Philadelphia section of the Mathematical Association of America will meet on December 2, 1939 at Bethlehem, Pennsylvania.

Scripta Mathematica, published at Amsterdam Avenue and 186th street in New York City, announces the publication of a collection of 36 plates to be used as visual aids in the teaching of mathematics. Portraits, curves and designs are included in the collection.

Dr. W. T. Puckett and Dr. P. G. Hoel have been added to the department of mathematics at the University of California at Los Angeles.

Dr. Arvid T. Lonseth (University of California) and Dr. George E. Hay (University of Toronto) have been appointed instructors at the Armour Institute of Technology in Chicago. Dr. John W. Calkin

(New Hampshire) has been appointed assistant professor at Armour Institute.

A meeting of the American Mathematical Society will be held at the University of Missouri on December 1-2, 1939. Professor E. W. Chittenden (University of Iowa) will deliver an invited address on *Topological functions*. Professor L. R. Ford (Armour Institute of Technology) will deliver an invited address on *Projective Transformations in two complex variables.*

The American Mathematical Society was scheduled to meet at Columbia University on October 28, 1939. Professor J. J. Gergen (Duke University) was scheduled to speak on *Double Fourier Series.*

The American Mathematical Society will hold its regular annual meeting in conjunction with the annual meeting of the Mathematical Association of America at Columbus, Ohio, on December 26-29, 1939. The Josiah Willard Gibbs Lecture will be delivered by Professor Theodore von Kármán (California Institute of Technology). His subject will be *The engineer grappling with non-linear problems.* Professor J. R. Kline, retiring chairman of Section A of the American Association for the advancement of science, will deliver an address on *The Jordan Curve theorem.*

The same lack of informed judgment that is apt to lead the high school student into a mistake in choosing his courses is just as often exhibited when he comes to decide, at the age of sixteen or seventeen, whether, or not, he will go to college or enter a vocation, upon leaving high school. We very seriously question the wisdom of having the secondary courses so organized that the high school graduate who did not choose the college-preparatory curriculum, must enter college gravely handicapped. Many have entered thus handicapped because their decision not to enter college was revised after leaving high school—but revised too late. Plainly, if the freshman year at college reveals to such a student a sufficient degree of mathematical ability to make him desire to major in that field, the handicap must then be disclosed. He chose no geometry in high school and cannot repair the omission by taking it in college since it is a secondary, not a college subject matter.—From *Mathematics News Letter,* volume VII, number 1.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

No. 272. Proposed by *Walter B. Clarke*, San Jose, California.

Given the line segments of lengths d, e, f. Locate these as three chords concurrent at P in a circle of minimum size so that P divides each chord as the orthocenter would divide it if they were altitudes of a triangle. What are the sides of the corresponding triangle?

Solution by *C. W. Trigg*, Los Angeles City College.

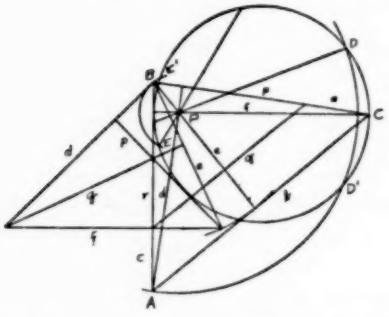
This problem is essentially one of constructing a triangle with sides a, b, c, given the corresponding altitudes d, e, f. To do this construct a triangle with sides d, e, f, and draw in the altitudes p, q, r. Construct a triangle with sides p, q, r. From vertex B drop the altitude to q and extend this to a length equal to e. Through its extremity draw a parallel to q, cutting p and r extended in C and A, respectively. The sides of triangle ABC are a, b, c and the altitudes which may be drawn are d, e, f, and intersect at P. Clearly $pd=qe=rf$ and since $ad=be=cf$, $p:q:r::a:b:c$.

In a given triangle the products of the two segments into which the orthocenter divides the altitudes are equal. So on the longest altitude, say f, as a diameter describe a circle, which is therefore the required minimum circle. With PB as radius and P as center describe an arc cutting the circle at E and E'. The chords through D and P, and through D' and P' will be equal to e, since the products of the segments of intersecting chords are equal. Likewise with PA as radius and P as center describe an arc meeting the circle at D and D'.

The chords through D and P and D' and P will equal d. Hence there are in general four solutions.

The sides of the corresponding triangle have been constructed. To express them in terms of the altitudes, note that the area of the triangle is

$$\tfrac{1}{2}ad=\sqrt{s(s-a)(s-b)(s-c)},\ \text{so}$$



$4a^2d^2=2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4$. Now $b=ad/e$ and $c=ad/f$. When these values are substituted in the aforesaid equation and the result is simplified, we have

$$a=2de^2f^2/\sqrt{2d^2e^2f^2(d^2+e^2+f^2)-d^4e^4-e^4f^4-f^4d^4}$$
$$=2de^2f^2/M.$$

Then by symmetry, $b=2ef^2d^2/M$ and $c=2fd^2e^2/M$.

No. 279. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

If a, b, c, each different from zero, is a solution of

$$px^3+qy^3+rz^3+sxyz=0,$$

where $27pqr+s^3\neq 0$, find a second solution, other than ka, kb, kc, expressed rationally in terms of a, b, c, and the coefficients.

Solution by the *Proposer*.

It is easily verified that if $(x, y, z)=(a, b, c)$ is a solution of the proposed equation, then

(1) $$x=a(rc^3-qb^3),\quad y=b(pa^3-rc^3),\quad z=c(qb^3-pa^3)$$

is also a solution. These values are not proportional to a, b, c: for (1) gives at once $x/a+y/b+z/c=0$ so that $x/a=y/b=z/c$ would imply $x=y=z=0$; then (since $abc\neq 0$) we would have $rc^3=qb^3=pa^3$, which would reduce the proposed equation to $3pa^3=-sabc$ or $3qb^3=-sabc$

or $3rc^3 = -sabc$, from which, by multiplication, we would have $27pqr = -s^3$, contrary to the hypothesis.

Usually repetitions of the transformation (1) will produce further solutions of the kind required by the problem.

EDITOR'S NOTE: The problem is complicated a little by the fact that the condition $abc \neq 0$ does not imply $xyz \neq 0$, and thus the solution given by (1) may fail to meet the requirements. Perhaps the hypothesis should contain the condition $pqr \neq 0$: otherwise two letters can be chosen at pleasure and the third determined from a linear equation. Then if it be merely demanded that the letters in a solution be not all zero, the difficulty is removed.

No. 280. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Find the volume of a tetrahedron in terms of its edges.

Solution by the *Proposer*.

In tetrahedron $S-ABC$ let $SS' = h$ be the altitude to the base ABC, whose sides are a, b, c, and draw $S'D$ and $S'E$ perpendicular to BC and AC respectively. Then SD and SE are perpendicular to BC and AC respectively. Set $SA = d$, $SB = e$, $SC = f$, $CD = x$, $CE = y$, $DE = z$.

Using triangles CED and ABC, we have

$$\cos C = (c^2 - a^2 - b^2)/2ab = (z^2 - x^2 - y^2)/2xy,$$

whence

$$z^2 = x^2 + y^2 - (a^2 + b^2 - c^2)xy/ab. \tag{1}$$

From the right triangles SDC, SDB,

$$f^2 - x^2 = e^2 - (BD)^2,$$

$$f^2 - e^2 = (x + BD)(x - BD),$$

whence $$x = (a^2 + f^2 - e^2)/2a.$$

Likewise $$y = (b^2 + f^2 - d^2)/2b.$$

Substituting in (1), we have:

$$4a^2b^2z^2 = b^2(a^2 + f^2 - e^2)^2 + a^2(b^2 + f^2 - d^2)^2 - (a^2 + b^2 - c^2)(a^2 + f^2 - e^2)(b^2 + f^2 - d^2). \tag{2}$$

If K and K' are the areas of triangles ABC and CDE, then $K'/K = xy/ab$. Since CS' is the diameter of the circumcircle of triangle CDE, $xyz = 2K'(CS')$. Hence $z = 2K(CS')/ab$ and

$$h^2 = f^2 - a^2b^2z^2/4K^2$$

and, substituting in (2):

$$16K^2h^2 = 16K^2f^2 - b^2(a^2+f^2-e^2)^2 - a^2(b^2+f^2-d^2)^2 + (a^2+b^2-c^2)(a^2+f^2-e^2)(b^2+f^2-d^2.)$$

If V is the volume of $S-ABC$,

$$16K^2h^2 = 144V^2 \text{ and } 16K^2 = -c^4+2(a^2+b^2)c^2-(a^2-b^2)^2.$$

Hence

$$144V^2 = [-c^4+2(a^2+b^2)c^2-(a^2-b^2)^2]f^2 - b^2(a^2+f^2-e^2)^2 - a^2(b^2+f^2-d^2)^2 + (a^2+b^2-c^2)(a^2+f^2-e^2)(b^2+f^2-d^2).$$

Also solved by *C. W. Trigg* who remarks: The same result was secured with the aid of spherical trigonometry by Trigg in School Science and Mathematics, March 1934, p. 316. It is essentially in the form given by Euler [Novi comment, acad. sc. Petrop., Vol. 4 (1752-3), 1758, p. 159]. W. Killing and H. Hovestadt (Handbuch des mathematischen Unterrichts, Leipzig, Vol. 2, 1913, p. 421) state this relation in the form,

$$144V^2 = (a^2+b^2+c^2+d^2+e^2+f^2)(a^2d^2+b^2e^2+c^2f^2) - 2a^2d^2(a^2+d^2) - 2b^2e^2(b^2+e^2) - 2c^2f^2(c^2+f^2) - a^2e^2f^2 - d^2b^2f^2 - c^2d^2e^2 - a^2b^2c^2.$$

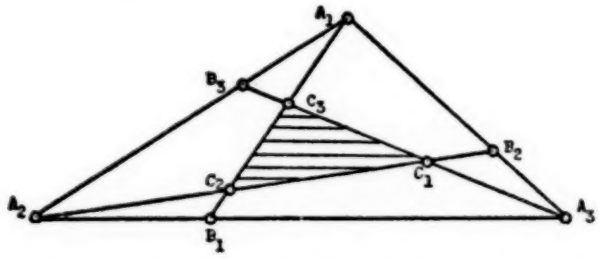
R. Baltzer (Theorie et Applications des Déterminants. Traduit de l'allemand. Paris, 1861, p. 206) gave the determinant form, which in the present notation is:

$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d^2 & e^2 & f^2 \\ 1 & d^2 & 0 & c^2 & b^2 \\ 1 & e^2 & c^2 & 0 & a^2 \\ 1 & f^2 & b^2 & a^2 & 0 \end{vmatrix}.$$

References to other expressions for the volume of a tetrahedron in terms of the edges and the radius of the circumscribed sphere are given in the American Mathematical Monthly, Vol. 28, pp. 279-81, (1921).

No. 282. Proposed by *M. Kac*, Lwów, Poland.

Given the triangle A_1,A_2,A_3, with B_1,B_2,B_3 as points trisecting the sides.



Let the intersections of the lines A_iB_i be C_1,C_2,C_3. Show that

$$7s = S$$

where s is the area of $C_1C_2C_3$ and S is the area of $A_1A_2A_3$.

Solution by *L. M. Kelly*, Boston University.

For triangle $A_1B_1A_3$ with transversal $B_2C_2A_2$, we have*

$$(A_2B_1)(A_3B_2)(C_2A_1)/(A_2A_3)(B_2A_1)(C_2B_1) = 1.$$

That is,

$$(1/3)(1/2)(C_2A_1)/(C_2B_1) = 1$$

or

$$(C_2B_1)/(B_1A_1) = 1/7.$$

Therefore

$$(A_2A_1C_2)/(A_2B_1A_1) = 6/7 \quad \text{or} \quad A_2C_2A_1 = (6/21)(ABC).$$

Similarly for triangles $A_1A_3C_3$ and $A_2A_3C_1$. Accordingly

$$C_1C_2C_3 = (1-18/21)(ABC) = (1/7)(ABC).$$

Solution by *D. L. MacKay*, Evander Childs High School, New York.

We may generalize the theorem by setting the ratios $A_2B_1 : B_1A_3$, $A_3B_2 : B_2A_1$, $A_1B_3 : B_3A_2$ equal respectively to m, n, p, where m, n, p are unequal.

The ratio of triangles $A_2C_1A_3$ and $A_1C_1A_3$ equals the ratio of their altitudes on C_1A_3. But the ratio of these altitudes equals $A_2B_3 : A_1B_3 = 1/p$. Hence $A_1C_1A_3 = (p)(A_2C_1A_3)$ and in like manner: $A_2C_1A_1 = (1/n)(A_2C_1A_3)$.

*By a simple application of the Law of Sines. See Ceva's Theorem.—ED.

Now $A_1A_2A_3 = A_3C_1A_1 + A_2C_1A_1 + A_2C_1A_3(p+1/n+1)(A_2C_1A_3).$

Hence $(A_2C_1A_3)/(A_1A_2A_3) = n/(1+n+np).$

Likewise, $(A_3C_3A_1)/(A_1A_2A_3) = p/(1+p+mp)$

$(A_1C_2A_2)/(A_1A_2A_3) = m/(1+m+mn).$

But $C_1C_2C_3 = A_1A_2A_3 - A_2C_1A_3 - A_3C_3A_1 - A_1C_2A_2$

$= [1 - n/(1+n+np) - p/(1+p+mp) - m/(1+m+mn)] \cdot (A_1A_2A_3)$

$= (A_1A_2A_3)(1-mnp)^2/(1+n+mp)(1+p+pm)(1+m+mn).$

For the special case of the problem, we take

$$m = n = p = 1/2.$$

Thus $C_1C_2C_3 = (1/7)(A_1A_2A_3).$

The problem appeared in Nouvelle correspondance mathematique, 1875, p. 105; 1876, p. 310.

C. W. Trigg remarks that the general theorem here provides a proof of the fact that the medians of a triangle are concurrent. That is, taking $m = n = p = 1$, we have: $C_1C_2C_3 = 0$.

In connection with these discussions, see an allied problem, No. 1580, in School Science and Mathematics, 39, No. 3, March, 1939, p. 282.

Also solved by *G. F. Alrich, Walter B. Clarke, Albert Farnell, G. W. Grotts, S. Haywood, Johannes Mahrenholz, J. Rosenbaum, Leo Simmons, P. D. Thomas* and *C. W. Trigg*.

No. 283. Proposed by *H. T. R. Aude*, Colgate University.

Find a sequence of triads of integers such that when taken as sides of triangles these will approach an isosceles triangle with an angle of 120°.

I. Solution by *E. C. Kennedy*, Texas College of Arts and Industries.

Let the triangle be isosceles with sides a, a, c. If angle C is 120°, c/a must equal $\sqrt{3}$, which is impossible in integers. Rational approximations as close to $\sqrt{3}$ as we please may be obtained as convergents to the development of $\sqrt{3}$ in a continued fraction, viz. $c/a = 1$, 2, 5/3, 7/4, 19/11, 26/15, etc. These are alternately less and greater than $\sqrt{3}$. Numerators and denominators of either the odd or the even series of convergents will satisfy the recursion formulæ

(1) $$a_{n+1} = 2a_n + c_n, \qquad c_{n+1} = 3a_n + 2c_n.$$

Thus there are obtained two sequences of isosceles triangles with angles approaching 120° from below and above, obtained from (1) by taking $a_1=3$, $c_1=5$ or $a_1=4$, $c_1=7$, respectively.

II. Solution by *C. W. Trigg*, Los Angeles City College.

If angle C is 120°, then $c^2=a^2+b^2+ab$, for which the parametric solution is known, viz. $a=m^2-n^2$, $b=2mn+n^2$, $c=m^2+mn+n^2$. If for successively increasing values of m, n, the difference between a and b remains constant, the ratio a/b approaches unity and the triangle becomes more nearly isosceles. Thus let $a-b=1$. The first two solutions of the resulting Pellian equation $(m-n)^2-3n^2=1$ are

$$(m-n,\ n)=(2,1) \text{ and } (7,4).$$

Subsequent values of $(m-n)$ and n may each be written from the recursion formula

$$u_{n+1}=4u_n-u_{n-1}.$$

In this manner we obtain as many triads as desired. The first few are: 7, 8, 13; 104, 105, 181; 1455, 1456, 2521; 20272, 20273, 35113.

Also solved by the *Proposer*.

No. 285. Proposed by *Albert Farnell*, Louisiana State University.

Prove that the sum of n terms of the series whose nth term is $n\cos(2an+b)$ is equal to

$$\frac{n\sin(2an+a+b)}{2\sin a}-\frac{\sin an\sin(an+b)}{2\sin^2 a}.$$

Solution by *Johannes Mahrenholz*, Cottbus, Germany.

From the known trigonometric identity

$$\sum_{x=1}^{n}\sin(2ax+b)=\sin an\sin(an+a+b)/\sin a,$$

by differentiation with respect to a, there results

$$\begin{aligned}\sum_{x=1}^{n}2x\cos(2ax+b)=[&n\sin a\cos an\sin(an+a+b)\\&+(n+1)\sin a\sin an\cos(an+a+b)\\&-\cos a\sin an\sin(an+a+b)]/\sin^2 a.\end{aligned}$$

The right member can be put in the form

$$\{n \sin a \sin[(an+a+b)+an]-\sin an \sin[(an+a+b)-a]\}/\sin^2 a,$$

from which the proposed formula follows immediately.

Also solved by *C. W. Trigg* and *M. E. Wescott.*

No. 287. Proposed by *Yudell Luke,* University of Illinois.

Let the digits of a 3-digit number N be reversed to form N'. Put D for the difference $|N-N'|$, and let the digits of D be reversed to form D'. (D is considered to have three digits: thus if $D=099$, $D'=990$). Then, if $D\neq 0$, $D+D'$ has the fixed value 1089. This well known fact is easily proved. What values may $D+D'$ take corresponding to N a 4-digit number?

Solution by *C. W. Trigg,* Los Angeles City College.

Let $N=abcd$. Noting that

$$(999)x=x-\overline{1}\,\overline{9}\,\overline{9}\,\overline{10-x} \quad \text{and} \quad (90)x=\overline{x-1}\,\overline{10-x}\,\overline{0},$$

where $0<x<10$ and the bars indicate separate digits, we may reduce the proposition to the following cases:

I. $(a=d,\ b=c)$, $D=0$, and $D+D'=0$;

II. $(a=d,\ b>c)$, $D=\overline{0}\,\overline{b-c-1}\,\overline{10+c-b}\,\overline{0}$, and $D+D'=0990$;

III. $(a>d,\ b>c)$, $D=\overline{a-d}\,\overline{b-c-1}\,\overline{10+c-b-1}\,\overline{10+d-a}$, and $D+D'=10890$;

IV. $(a>d,\ b=c)$, $D=\overline{a-d-1}\,\overline{9}\,\overline{9}\,\overline{10+d-a}$, and $D+D'=10989$;

V. $(a>d,\ b<c)$, $D=\overline{a-d-1}\,\overline{10+b-c}\,\overline{c-b-1}\,\overline{10+d-a}$, and $D+D'=9999$.

All numbers not given above are brought under cases II-V by taking $abcd$ as N' instead of N.

Also solved by *Albert Farnell* and the *Proposer.*

No. 289. Proposed by *V. Thébault,* Le Mans, France.

In the equation

$$X^2=Y^2+Z^2 \quad \text{and} \quad x^2=y^2+z^2$$

the letters represent positive integers, and Y and Z are relatively prime, as also y and z. Show that the numbers

$$Xx+Yy+Zz \quad \text{and} \quad Xx+Yz+Zy$$

are, the one a square, the other twice a square.

Solution by *Annie Christensen*, Baton Rouge, La.

By a familiar analysis the integers which satisfy the given equations are

$$X=M^2+N^2, \qquad Y=2MN, \qquad Z=M^2-N^2,$$
$$x=m^2+n^2, \qquad y=2mn, \qquad z=m^2-n^2.$$

From these values we have at once

$$Xx+Yy+Zz=2(Mm+Nn)^2 \quad \text{and}$$
$$Xx+Yz+Zy=(Mm+Nm+Mn-Nn)^2.$$

The permissible interchange of Y and Z (or of y and z) is seen merely to interchange these results.

C. W. Trigg notes that the condition of relative primality is not necessary to the conclusion. Also solved by *H. T. R. Aude*, *L. M. Kelly*, *Johannes Mahrenholz* and the *Proposer*.

No. 295. Proposed by *V. Thébault*, Le Mans, France.

In the system of numeration with base 31, form all perfect squares of four digits having the form *aabb*. What notable properties appear in certain of these?

Solution by *C. W. Trigg*, Los Angeles City College.

Put $$N^2=aabb=(r+1)(ar^2+b)=32(961a+b)=(8P)^2,$$

so that $$P^2=480a+\tfrac{1}{2}(a+b).$$

Since neither a nor b may exceed 30, we may quickly obtain all possible values of P^2. Thus there are nine solutions, namely, $(5\ \overline{21})^2=1\ 1\ 7\ 7$, $(8\ 0)^2=2\ 2\ 0\ 0$, $(9\ \overline{25})^2=3\ 3\ 5\ 5$, $(\overline{11}\ \overline{11})^2=4\ 4\ \overline{28}\ \overline{28}$, $(\overline{14}\ \overline{30})^2=7\ 7\ 1\ 1$, $(\overline{16}\ 0)^2=8\ 8\ 0\ 0$, $(\overline{19}\ \overline{19})^2=\overline{12}\ \overline{12}\ \overline{20}\ \overline{20}$, $(\overline{24}\ 0)^2=\overline{18}\ \overline{18}\ 0\ 0$, $(\overline{29}\ \overline{29})^2=\overline{28}\ \overline{28}\ 4\ 4$.

Solved also by *Johannes Mahrenholz* and the *Proposer*, who (perhaps to forestall the use of this as a source for further problems) notes three solutions of $aabb=(cc)^2$, a solution of $aabb=x^2$ and $bbaa=y^2$, and finally a solution of $aabb=(cc)^2$ and $bbaa=(dd)^2$.

PROPOSALS

No. 318. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

If $$\cot A+\cot B+\cot C=\sqrt{3},$$

then triangle ABC is equilateral.

No. 319. Proposed by *V. Thébault*, Le Mans, France.

Determine a system of numeration in which a four-digit number of the form $aaab$ is the square of a two-digit number cd, having given that a and b are consecutive digits, as are also c and d.

No. 320. Proposed by *H. T. R. Aude*, Colgate University.

In a triangle ABC the three altitudes are drawn meeting in the point H. The feet of the altitudes are, respectively, D, E, F. On the line segments DH, EH, and FH as diameters circles are drawn. Show that the three common chords of these three pairs of circles are equal. Also show that they are perpendicular, respectively, to the sides of the triangle DEF.

No. 321. Proposed by *A. A. Aucoin* and *W. V. Parker*, Louisiana State University.

Determine a five-parameter solution for

$$(x+2y+z)(x+y)(y+z)(x+2z)=u^2+2uv+5v^2.$$

No. 322. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

Find the locus of the point which moves so that its power for a given sphere bears a constant ratio to its distance from a given plane.

No. 323. Proposed by *W. C. Janes*, Lincoln, Nebraska.

Verify the following identities:

(1) $$2^{2n-1}\sum_{r=0}^{n-1}\frac{(2n-r-1)\cdots(2n-2r)}{r!}$$

$$=\sum_{r=0}^{n-1}\frac{2n\cdots(2n-2r)}{(2r+1)!}\,5^r;$$

$$(2) \qquad 2^{2n}\sum_{r=0}^{n}\frac{(2n-r)\cdots(2n-2r+1)}{r!} = \sum_{r=0}^{n}\frac{(2n+1)\cdots(2n-2r+1)}{(2r+1)!}\ 5'.$$

No. 324. Proposed by *Walter B. Clarke*, San Jose, California.

Given a point P and two intersecting lines, all in a plane. (1) Construct two equal and perpendicular line segments from P to the given lines. (2) Let Q be the vertex that lies opposite P of a square on the segments as adjacent sides. What is the locus of Q as P describes a line?

Errors noted Vol. XIII, No. 8, May, 1939.

p. 386, line 7; for 3 read $\sqrt{3}$.

p. 389, line 29; insert $=$ between $(mn)^2$ and $abcd$.

last line; for 3021 read 3025.

Herbert Spencer, founder of the "System of Synthetic Philosophy," the most gigantic intellectual product of the 19th century, like many others of polyphase genius was mathematician before he was philosopher. Apparently, with no interest in pursuing an academic degree, he yet devoted the leisure time of his 'teens and early twenties to mathematics and the work of a civil engineer. One easily conjectures from a study of his development that in early manhood his faculties craved the unlimited athletics to be found in mathematics, that as his intellect further evolved it passed to engineering or other applications of mathematical science, from which in a final evolution it emerged to embrace the problem of problems—one whose solution would furnish a generalization of the "complete history of the knowable universe."—From *Mathematics News Letter*, volume VIII, number 6.

Bibliography and Reviews

Edited by
H. A. SIMMONS

Distance Geometries, a Study of the Development of Abstract Metrices. By Leonard M. Blumenthal. University of Missouri Studies, 145 pages, Vol. XIII, No. 2 (April, 1938). Price $1.25.

The class of readers for whom this book is intended is not well defined. The aims of the book are set forth in the introduction, and they are "(1) to give a critical survey of the material available concerning certain distance geometries..., (2) to correct certain errors and inaccuracies which have hitherto escaped attention, (3) to present a hitherto unpublished solution of a problem of importance in the metric geometry of a sphere..., (4) to discuss some of the applications of metric geometry..., (5) to indicate some unsolved problems..., and to point out areas awaiting development."

The book deals with spaces with a metric; that is, spaces consisting of points, which are undefined, and with a non-negative real number, pq, distance, attached to each pair of points p and q. This number must satisfy the following postulates:

I. $pq>0$ if p is different from q.
II. $pq=0$ if p and q are identical.
III. $pq=qp$.
IV. For every set of three points p,q, and r it is true that $pq+qr\geq pr$.

A space in which only the first three of these postulates are known to be satisfied is called semi-metric.

The general idea of the subject of distance geometry is to decide how much a specific distance function tells about the space in which it is defined. A typical result of the subject is Karl Menger's theorem (p. 57 of the book) that if there is some number n such that corresponding to every set of $(n+3)$ points of the space under consideration there can be found a set of $(n+3)$ points in an n-dimensional euclidean space such that the distances between the pairs of points of the euclidean are the same as those in the original space, then there exists in the euclidean space a set of points with one point corresponding to each point of the original space such that the distance between pairs of points in the euclidean space is the same as that between correspond-

ing pairs of points in the given space. It is easy to see that not every metric space has this property by constructing a set of four points p,q,r and s, and setting $pq = qr = rp = 1$ and $ps = qs = rs$, with the common value of this last set of distances any number greater than $1/2$ and less than $1/\sqrt{3}$, which is two-thirds the length of the median of an equilateral triangle with unit sides. This set of points can obviously not be embedded in euclidean space of any number of dimensions.

This example illustrates the difficulty in deciding what classes of people would want to read this book. Certainly the principles on which the geometry is founded are simple, and usually the results are simple. However, somewhere in between the announcement of the postulates and the announcement of the results there often lies some involved argument. To understand Menger's theorem stated above is easy; to prove it is another matter. Proofs are largely omitted in the book, but this does not necessarily help matters. The book itself seems mainly devoted to abstracts of the papers listed in the bibliography with critical and explanatory notes by the author. The hundred papers listed in the bibliography are covered in 175 pages. Sometimes the amount of compression necessarily leads to difficulty in reading; this seems especially true in the first chapter where, following Professor Menger's somewhat flowery introduction, the author with considerable abruptness disposes of the relations between distance geometries and more general geometries (relations which do not interest him in this book) and some of the early results of distance geometries. By the second chapter, however, he settles down to the business at hand, and those who survived or skipped the first chapter find a not particularly hurried exposition of the metric theory of convexity. In the third chapter he sets himself the task of solving specific problems, mainly problems of embedding, and thus arrives at statements concerning the relations between distance geometries and less general geometries. In all this work determinants play an important role, and in the fourth chapter the author considers some theorems which were proved without recourse to the theory of determinants; under the principle that many theorems of distance geometry are equivalent to theorems concerning determinants he reverses the more general procedure and proves the determinant theorems from the geometric ones. At this stage the reader who is not an expert in the field will probably profit by reading the first chapter again. The next two chapters are devoted to differential geometry, and the general problem is to determine how far a purely metric treatment independent of such things as coordinate systems can go in ordinary differential geometry. The transition here is one from a simultaneous consideration of all

points of the space to a consideration of the geometry near a given point. The resuls here are illuminating to anyone at all familiar with differential geometry. Notions of length and of curvature are introduced and exploited to advantage. The final chapter contains a few odds and ends which are for the most part instructive. There is a collection of "paradoxes" concerning measure, based on plausible-sounding statements known to be false; they are interesting but not particularly surprising even when accompanied by romantic statements replete with exclamation points, like the one in the middle of page 135 due to Menger. However, statements which sound plausible and are known to be false are in many ways more instructive than the more usual type of theorem, and though the advertisement of these things as paradoxes seems fraudulent, still they seem to merit their position in this book.

This book is an important one for the expert, for it makes a definite attempt to be complete in discussing the results of many branches of the subject. It should be interesting to many others because of the fundamental nature of the subject and its aliveness. The book is already out of date; many papers on the subjects have appeared since the bibliography was completed; the unpublished results of Scheenberg announced on page 85 have appeared in volume 44, number 3, pages 522-536 of the Transactions of the American Mathematical Society (November 1938) and in volume 39, number 4, pages 811-841 of the Annals of Mathematics (October, 1938). Karl Menger who has been the principal contributor to the theory recently, and who was mainly responsible for drawing the author of this book into the field, is still attracting new talent, so it seems likely that the field will be a lively one for sometime to come. Thus the book contains for the general mathematician an adequate survey of one of the really unifying branches of contemporary mathematics, and it throws in as an example a new solution of a representative problem in the field (beginning on page 76), and from time to time suggests many problems, some esay and some not easy, whose solutions would be of interest.

If this review has described the subject rather than the book, it is due to the fact that the book consists of the subject presented in a straightforward way, without any serious attempt at embellishing it.

Princeton University. C. B. TOMPKINS.

Analytic Geometry and Calculus, New Edition. By Woods and Bailey. Ginn and Co. 1938. xi+487 pages.

This book is a new edition of a work that was a revision and abridgment of the author's "Course in Mathematics for Students of Engineering and Science."

The first 129 pages are devoted to plane analytic geometry, pages 130-299 to the calculus, pages 300-334 to solid analytic geometry, pages 335-404 to partial differentiation and multiple integrals, then there is a chapter on infinite series followed by a chapter on differential equations.

There is ample material for a two year course.

The material is arranged so that all of the analytic geometry can be taken first and then the calculus if the teacher so desires.

The analytic geometry contains a very good selection of material and there is an unusually large number of well selected illustrative examples. There are a few minor corrections that should be made. On page 9 the definition of a variable should be given in terms of a symbol rather than a quantity. On page 26 the constant c must be different from zero in order that all of the conditions stated be satisfied. Page 53 line 7 should read "and may be any positive number except 1."

The calculus is presented from the point of view that, in an introductory course, emphasis should be placed upon gaining familiarity with the methods involved and the ability to solve problems. From this point of view the material is well selected and well arranged. The early introduction of the definite integral should help the student to realize that the calculus is a very powerful instrument for solving certain types of problems.

There is an excellent selection of well graded problems at the end of each chapter.

For the teacher who is interested in presenting clear cut definitions and from these deriving theorems by pure logic, that part of the book dealing with the calculus will not be entirely satisfactory. The definition of the limit of a variable on page 130 and the definition of continuity on page 136 do not present clear cut ideas. The author's definition of continuity implies only one condition whereas there are two conditions to be satisfied when a function is continuous at a point. In several places theorems are stated as obvious or evident facts. (See page 144 line 13 and page 149 line 4 from the bottom.) The proofs of some of the theorems rest upon intuition rather than logic. The inequality on page 146 line 12 is not necessarily true in all cases. In several places the authors merely prove that if a function has a

derivative then it is equal to a certain expression, (cf. pages 197, 198, 291, 345). There is no reason for assuming the existence of the derivative when there are elementary proofs that the derivative actually exists.

The early introduction of the mean value theorem and at least a statement of a modified form of Duhamel's theorem would provide the means of giving logical proofs of a good many of the theorems in the calculus.

Heidelberg College. JESSE PIERCE.

Vector Analysis. By J. H. Taylor. Prentice-Hall, 1939, New York. ix+177 pages.

This book devotes 63 pages to the algebra of vectors, 37 pages to the differential calculus of vectors, 36 pages to the integral calculus of vectors, and 36 pages to an introduction to tensor analysis. The author acknowledges special indebtedness to the works of Veblen, Weyl, Gibbs-Wilson, Lunn, and E. H. Moore, as a basis for his book, and gives at the end a list of fifty-four books which are used as references in the text.

Vectors are designated by bold type letters, and products are indicated by the dot and the cross, following Gibbs.

The book shows the influence of modern higher algebra in its explicit formulation of assumptions or definitions regarding a coordinate system, a group, linear spaces, linear dependence, basis of a linear vector space, and so on.

The abstract theory is supplemented by illustrations of its application to the geometry of curves and surfaces, the motions of rigid bodies, the flow of fluids, and the flow of heat. Exercises for the student are fairly numerous, many of them theoretical and fairly difficult.

The general typography is excellent, the pages being clear and attractive, with important headings or words standing out in bold type, rendering the skeleton of the subject easy to see and references easy to find.

The book should prove a useful addition to the list of earlier works on the same subject.

Northwestern University. E. J. MOULTON.